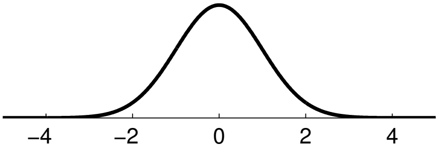
Chapter 3

- 3.1 Marginalization
- 3.2 Normal distribution with a noninformative prior (important)
- 3.3 Normal distribution with a conjugate prior (important)
- 3.4 Multinomial model (can be skipped)
- 3.5 Multivariate normal with known variance (useful for chapter 4)
- 3.6 Multivariate normal with unknown variance (glance through)
- 3.7 Bioassay example (very important, related to one of the exercises)
- 3.8 Summary (summary)

Normal / Gaussian

$$p(y|\mu) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\mu)^2\right)$$



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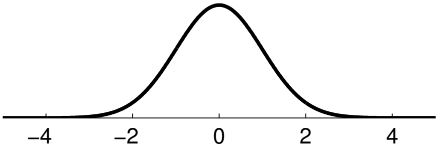

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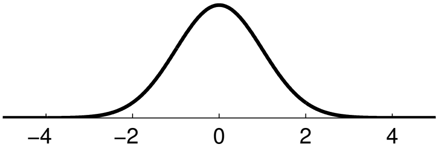

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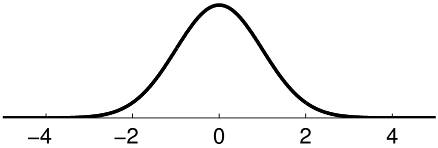

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- Gaussian: favored by Germans (but it was studied earlier by De Moivre and Laplace, and Gauss avoided using it)
- Shorthand notation
  - $y \sim \mathrm{N}(\mu, \sigma^2)$ with variance σ^2 (useful in derivations)
  - $y \sim \mathtt{normal}(\mu, \sigma)$ with deviation σ (useful for interpreting prior and posterior scales, used in Stan)

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- Normal linear regression
  connection to least squares regression via $(y - \mu)^2$

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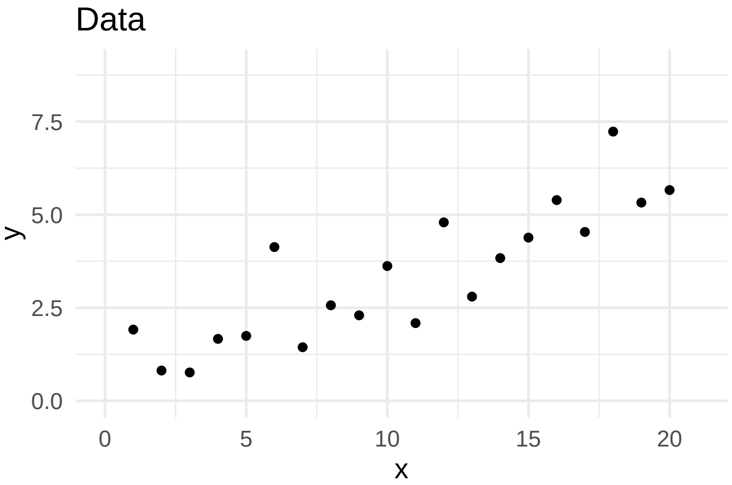
- Posterior distribution approximation with Laplace, variational inference, expectation propagation

Example of uncertainty in modeling



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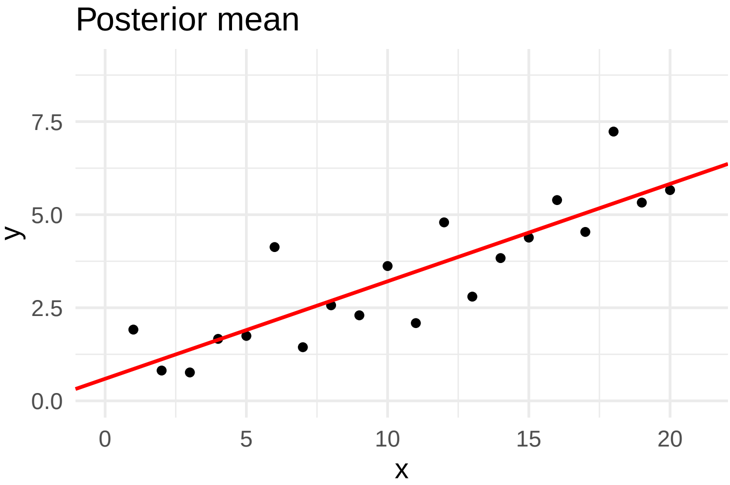

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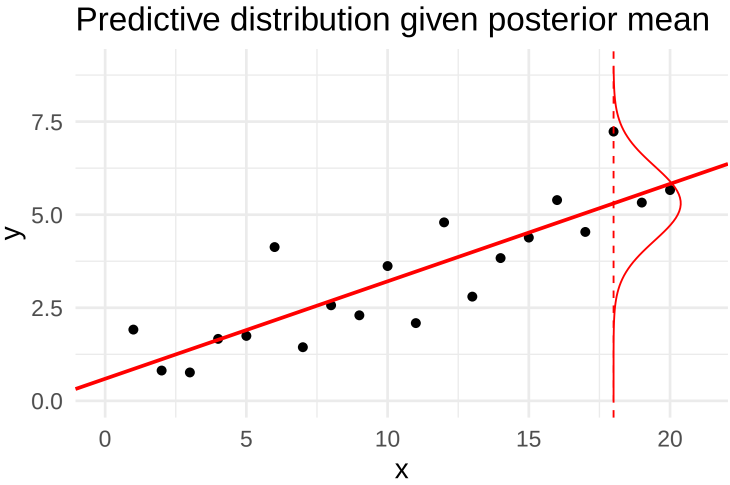

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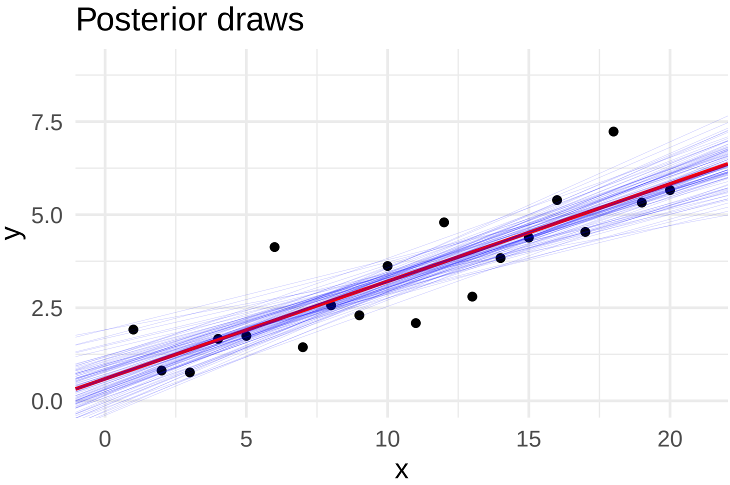

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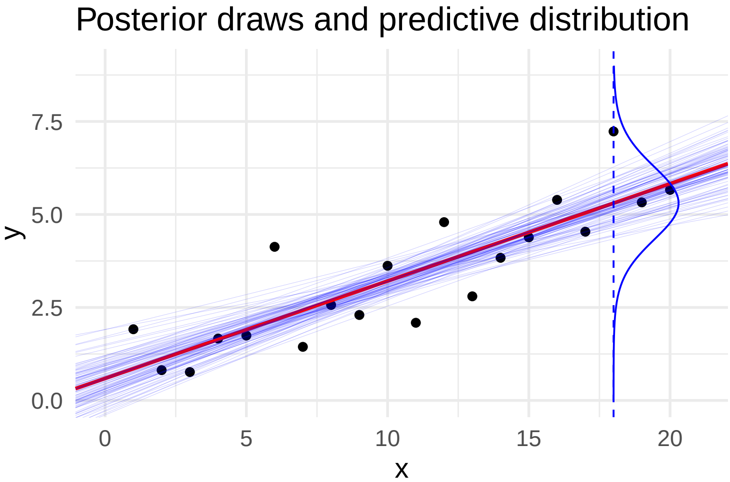
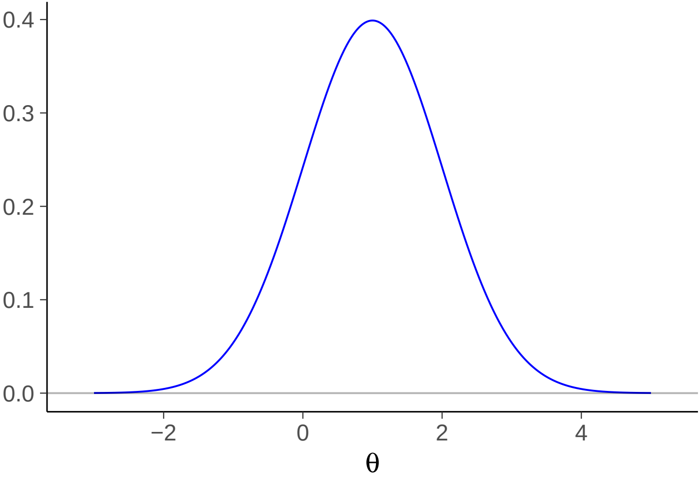


Monte Carlo and posterior draws

Density $p(\theta|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(\theta-\mu)^2\right)$



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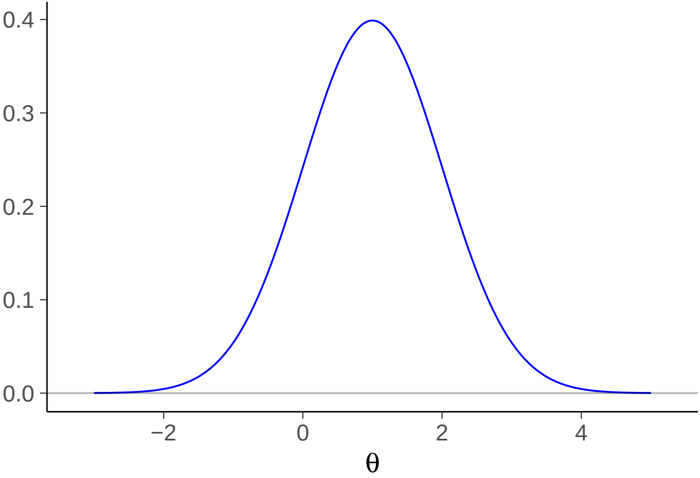

$\mathrm{E}(\theta) = \int \theta p(\theta|\mu,\sigma) d\theta = \mu$

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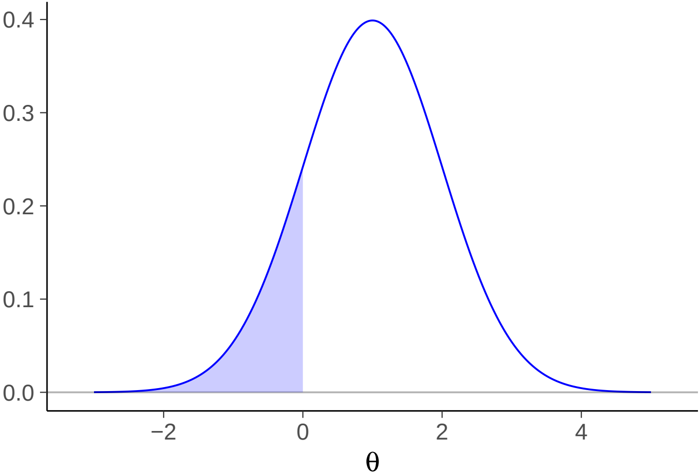
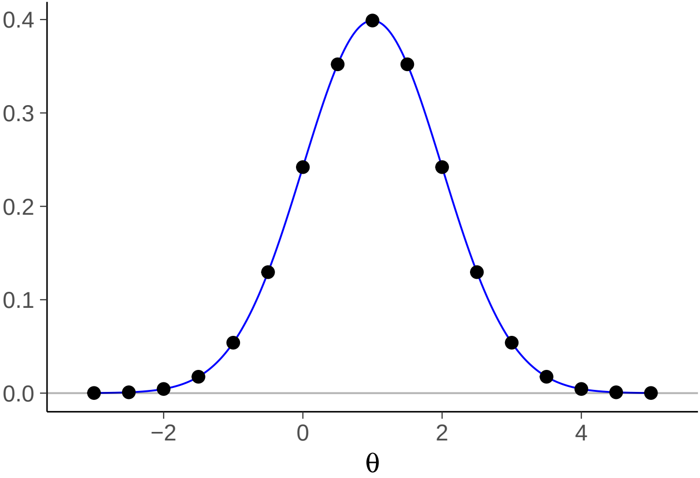


$p(\theta \leq 0) = \int_{-\infty}^{0} p(\theta|\mu,\sigma) d\theta$, many numerical approximations

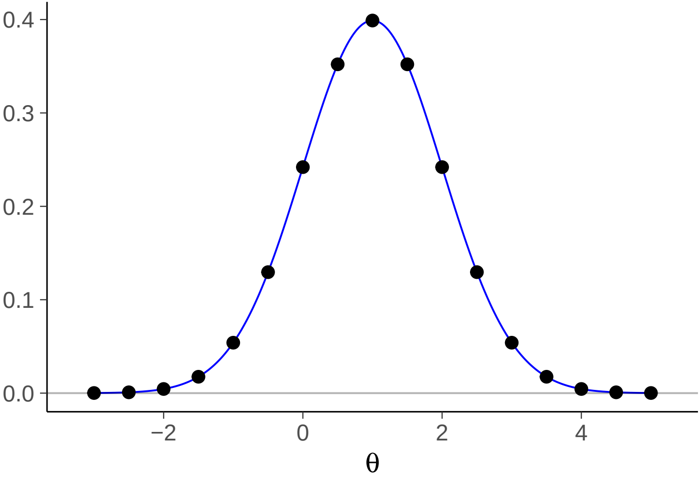
Monte Carlo and posterior draws

In practice evaluate in finite number of locations



Monte Carlo and posterior draws

Here evaluated in grid with bin width 0.5



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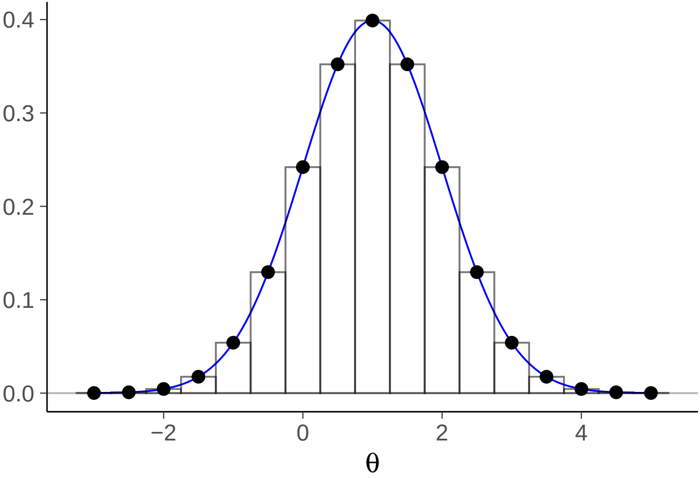

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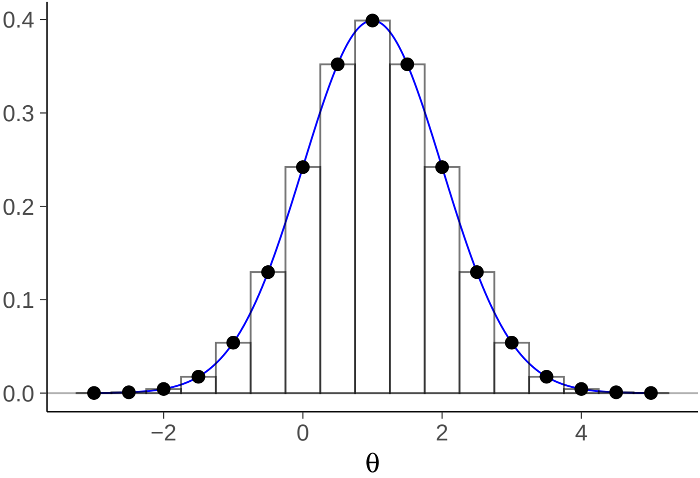
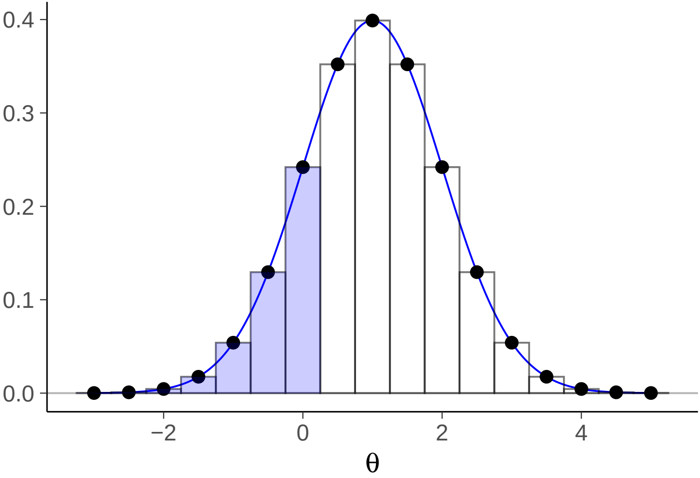


$$\mathrm{E}(\theta) = \int \theta p(\theta) d\theta \approx \sum_s^S \theta^{(s)} w_s \approx 1, \text{ where } w_s = 0.5 p(\theta)$$

Monte Carlo and posterior draws

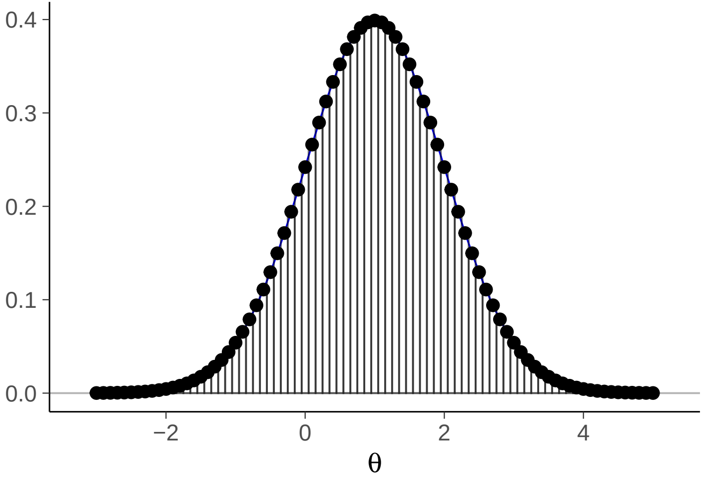
Here evaluated in grid with bin width 0.5



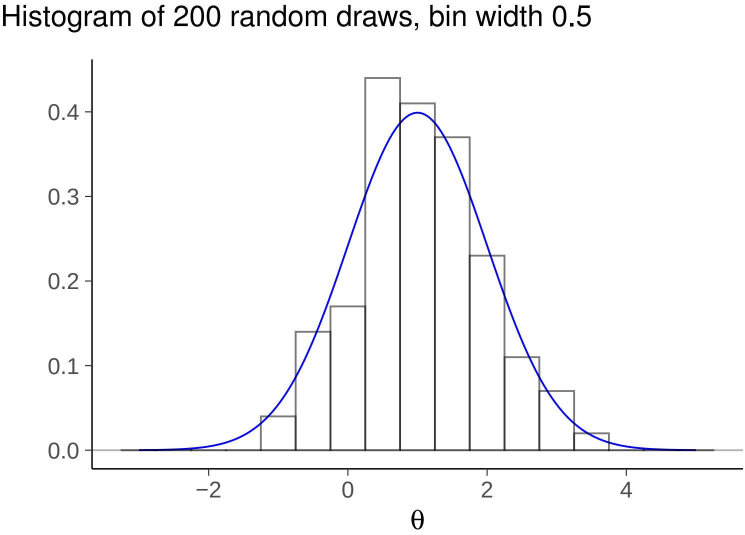
$p(\theta \leq 0) = \int_{-\infty}^{0} p(\theta) d\theta \approx \sum_{s}^{S} \mathrm{I}(\theta^{(s)} \leq 0) w_s \approx 0.22$

Monte Carlo and posterior draws

Here evaluated in grid with bin width 0.1

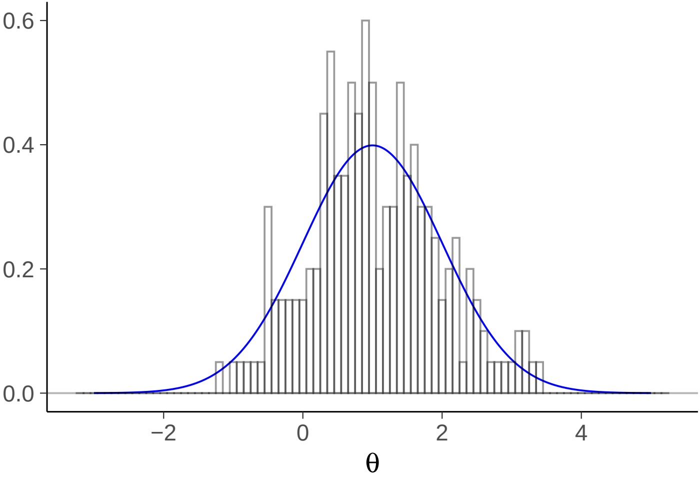


Monte Carlo and posterior draws



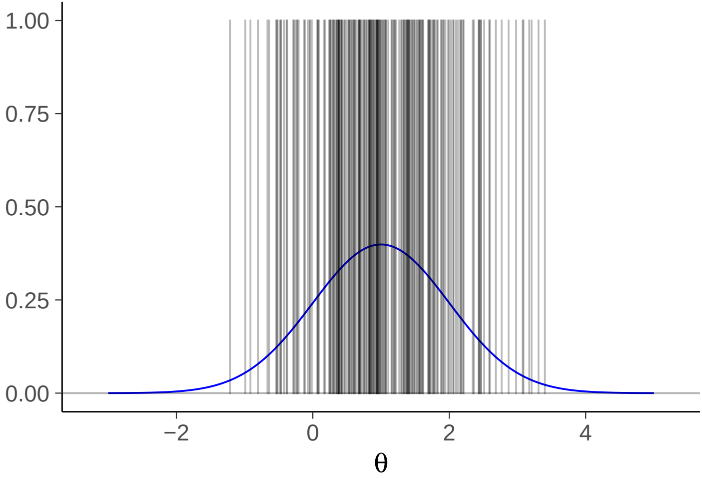
Monte Carlo and posterior draws

Histogram of 200 random draws, bin width 0.1



Monte Carlo and posterior draws

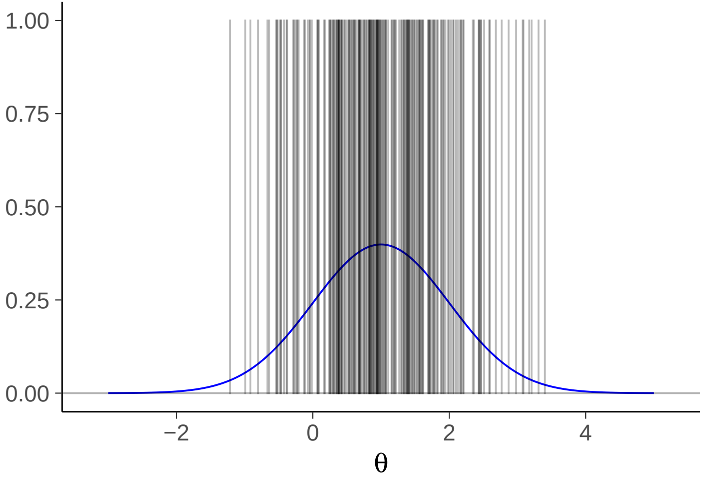
Histogram of 200 random draws, bin width 0



each bin has either 0 or 1 draw (and 0's can be ignored)

Monte Carlo and posterior draws

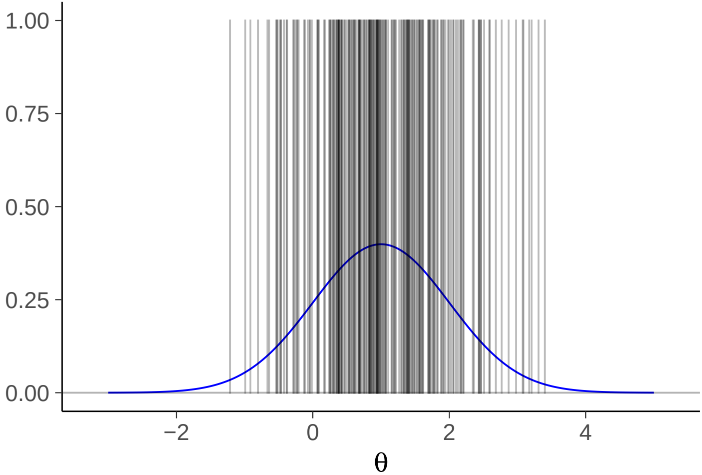
Histogram of 200 random draws, bin width 0



each bin with 1 draw has weight $1/S$

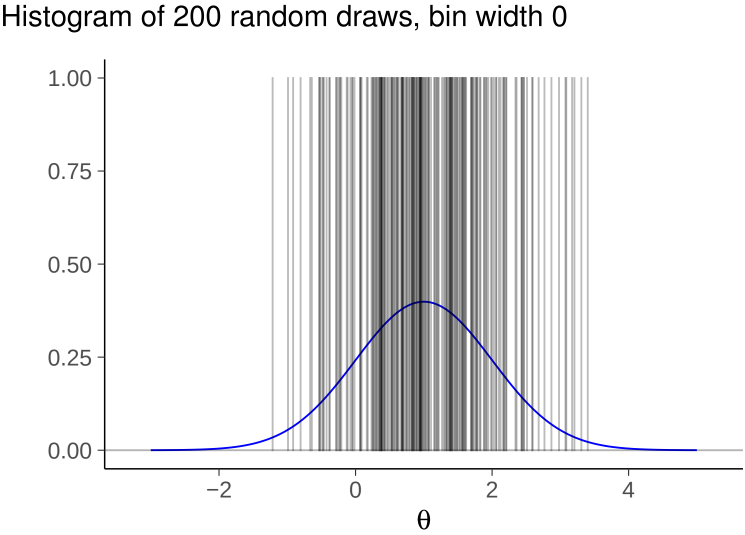
Monte Carlo and posterior draws

Histogram of 200 random draws, bin width 0



$E(\theta) \approx \frac{1}{S} \sum_{s}^{S} \theta^{(s)} \approx 1$

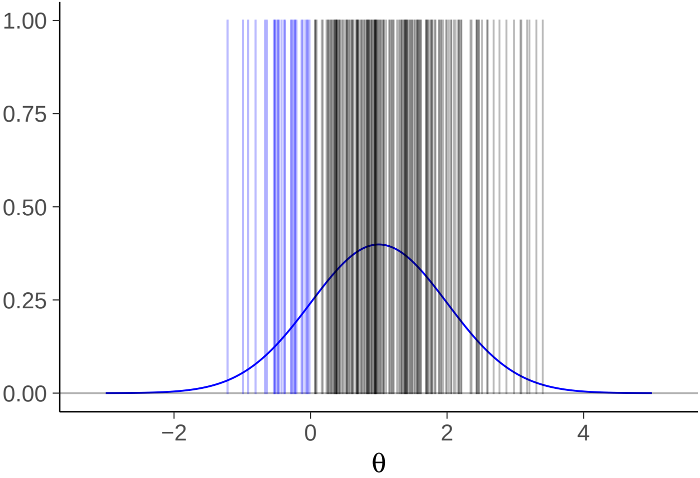
Monte Carlo and posterior draws



$E(\theta) \approx \frac{1}{S}\sum_s^S \theta^{(s)} \approx 1$, Monte Carlo estimate

Monte Carlo and posterior draws

Histogram of 200 random draws, bin width 0



$p(\theta \leq 0) \approx \frac{1}{S} \sum_{s}^{S} \mathrm{I}(\theta^{(s)} \leq 0) \approx 0.14$

Monte Carlo and posterior draws

- $\theta^{(s)}$ draws from $p(\theta \mid y)$ can be used
  - for visualization

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  - to approximate expectations (integrals)

$$E_{p(\theta|y)}[\theta] = \int \theta p(\theta \mid y) \approx \frac{1}{S} \sum_{s=1}^{S} \theta^{(s)}$$

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$$E_{p(\theta|y)}[\theta] = \int \theta p(\theta \mid y) \approx \frac{1}{S}\sum_{s=1}^{S} \theta^{(s)}$$

  - easy to approximate expectations of functions

$$E_{p(\theta|y)}[g(\theta)] = \int g(\theta) p(\theta \mid y) \approx \frac{1}{S}\sum_{s=1}^{S} g(\theta^{(s)})$$

Marginalization

- Joint distribution of parameters

$$p(\theta_1, \theta_2 \mid y) \propto p(y \mid \theta_1, \theta_2) p(\theta_1, \theta_2)$$

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- Monte Carlo approximation

$$p(\theta_1 \mid y) \approx \frac{1}{S} \sum_{s=1}^{S} p(\theta_1 \mid \theta_2^{(s)}, y),$$

where $\theta_2^{(s)}$ are draws from $p(\theta_2 \mid y)$

Marginalization - predictive distribution

- Posterior predictive distribution is obtained by marginalizing out the posterior distribution

$$p(\tilde{y} \mid y) = \int p(\tilde{y}, \theta \mid y) d\theta$$

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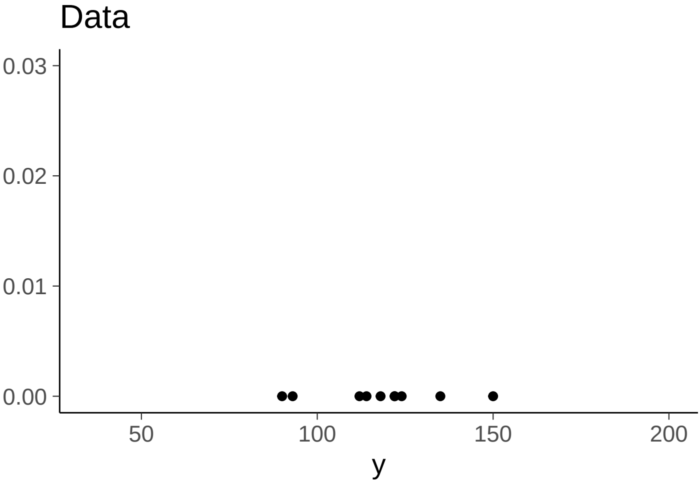
$$
\begin{aligned}
p(\tilde{y} \mid y) &= \int p(\tilde{y}, \theta \mid y) d\theta \\
&= \int p(\tilde{y} \mid \theta) p(\theta \mid y) d\theta
\end{aligned}
$$

Normal distribution example



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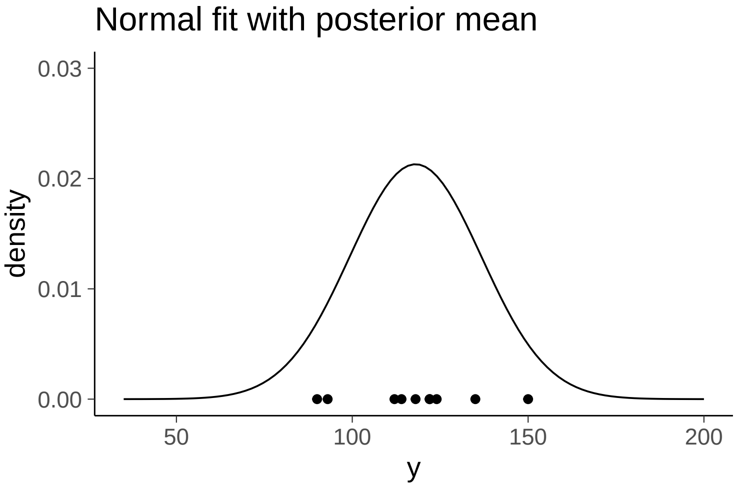
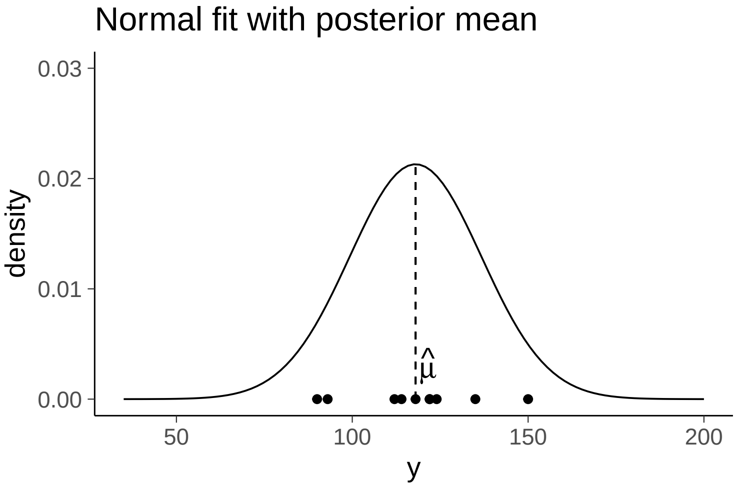


$$p(\textcolor{red}{y} \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(\textcolor{red}{y} - \mu)^2\right)$$

Normal distribution example



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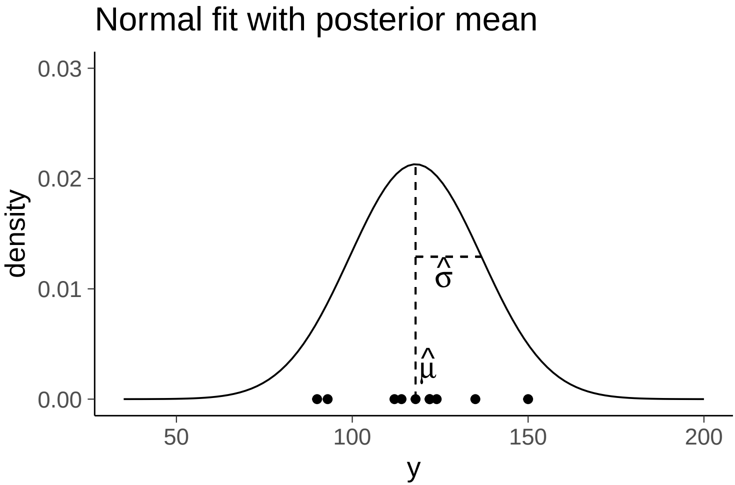

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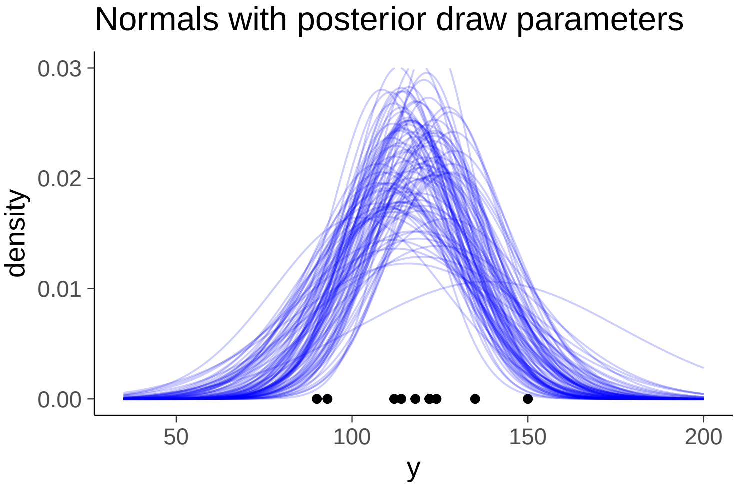
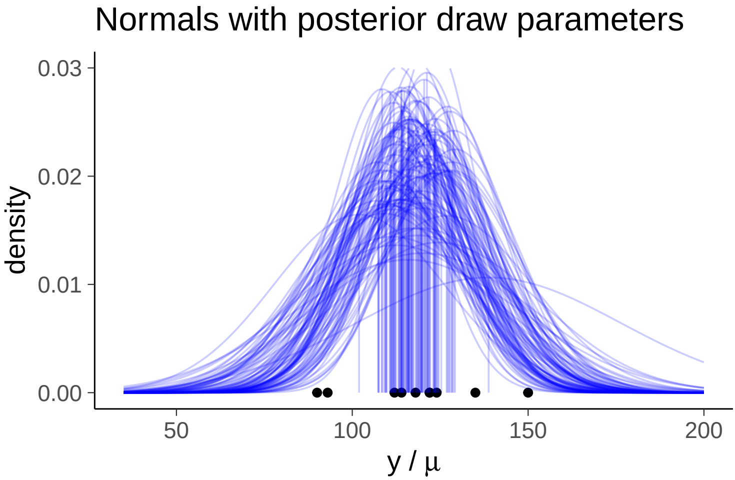


$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

Normal distribution example



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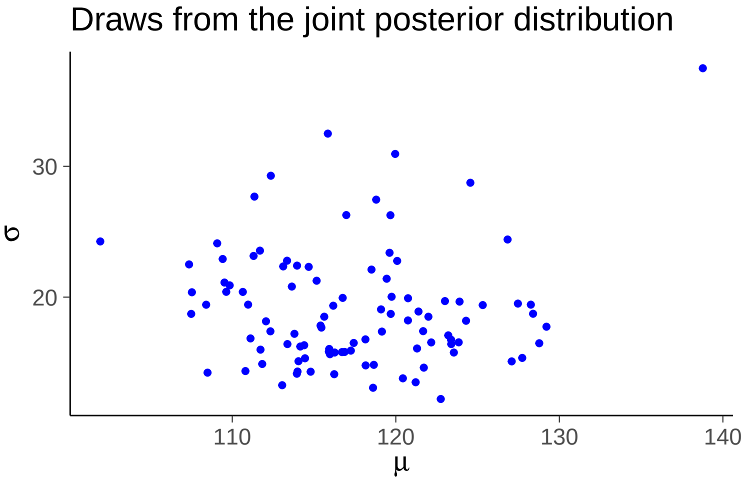

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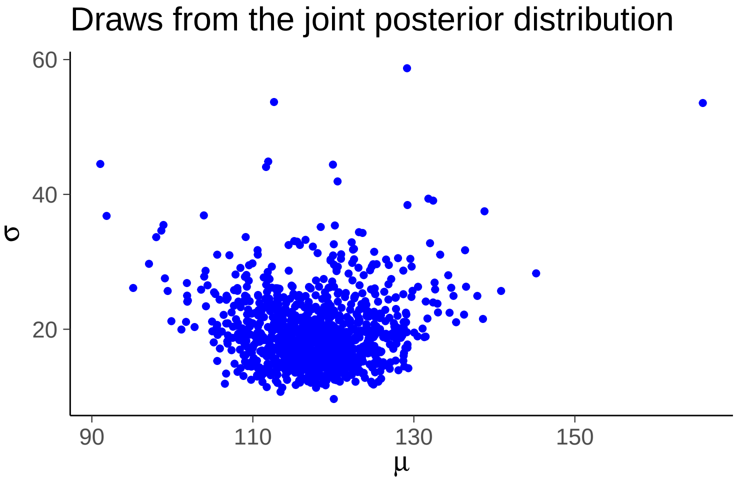

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Joint posterior

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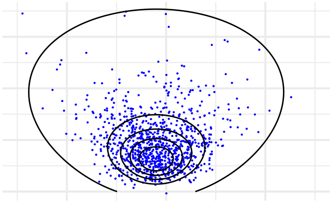

Joint posterior

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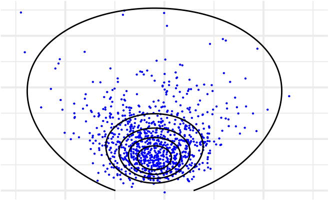
with $p(\mu, \sigma^2) \propto \sigma^{-2}$



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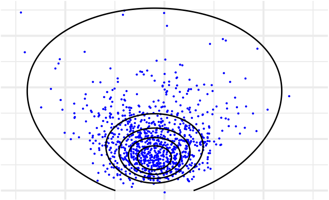
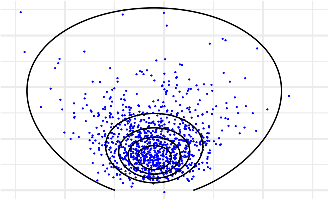


with $p(\mu, \sigma) \propto \sigma^{-1}$ (see BDA3 p. 21 transformation of variables)

Joint posterior



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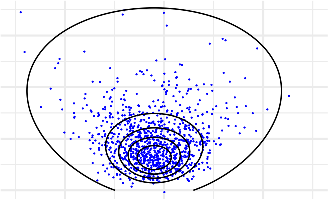
with $p(\mu, \sigma^2) \propto \sigma^{-2}$

$$p(\mu, \sigma^2 \mid y) \propto \sigma^{-2} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y_i - \mu)^2\right)$$

Joint posterior



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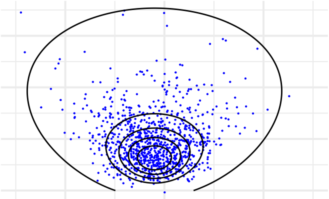
$$p(\mu, \sigma^2 \mid y) \propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \mu)^2\right)$$

Joint posterior



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$$p(\mu, \sigma^2 \mid y) \propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \mu)^2\right)$$

$$= \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(y_i - \bar{y})^2 + n(\bar{y} - \mu)^2\right]\right)$$

where $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$

Joint posterior

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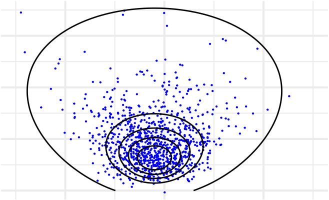
with $p(\mu, \sigma^2) \propto \sigma^{-2}$



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where $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$

$$= \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y} - \mu)^2\right]\right)$$

where $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$

Normal - non-informative prior

$$\sum_{i=1}^{n}(y_i - \mu)^2$$

Normal - non-informative prior

$$\sum_{i=1}^{n}(y_i - \mu)^2$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2)$$

Normal - non-informative prior

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$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2)$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2 - \bar{y}^2 + \bar{y}^2 - 2y_i\bar{y} + 2y_i\bar{y})$$

Normal - non-informative prior

$$\sum_{i=1}^{n}(y_i - \mu)^2$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2)$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2 - \bar{y}^2 + \bar{y}^2 - 2y_i\bar{y} + 2y_i\bar{y})$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\bar{y} + \bar{y}^2) + \sum_{i=1}^{n}(\mu^2 - 2y_i\mu - \bar{y}^2 + 2y_i\bar{y})$$

Normal - non-informative prior

$$\sum_{i=1}^{n}(y_i - \mu)^2$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2)$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2 - \bar{y}^2 + \bar{y}^2 - 2y_i\bar{y} + 2y_i\bar{y})$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\bar{y} + \bar{y}^2) + \sum_{i=1}^{n}(\mu^2 - 2y_i\mu - \bar{y}^2 + 2y_i\bar{y})$$

$$\sum_{i=1}^{n}(y_i - \bar{y})^2 + n(\mu^2 - 2\bar{y}\mu - \bar{y}^2 + 2\bar{y}\bar{y})$$

Normal - non-informative prior

$$\sum_{i=1}^{n}(y_i-\mu)^2$$

$$\sum_{i=1}^{n}(y_i^2-2y_i\mu+\mu^2)$$

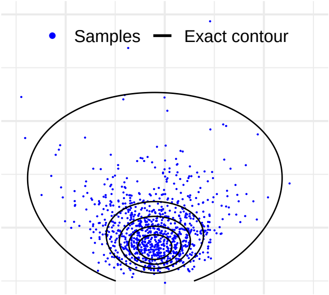
$$\sum_{i=1}^{n}(y_i^2-2y_i\mu+\mu^2-\bar{y}^2+\bar{y}^2-2y_i\bar{y}+2y_i\bar{y})$$

$$\sum_{i=1}^{n}(y_i^2-2y_i\bar{y}+\bar{y}^2)+\sum_{i=1}^{n}(\mu^2-2y_i\mu-\bar{y}^2+2y_i\bar{y})$$

$$\sum_{i=1}^{n}(y_i-\bar{y})^2+n(\mu^2-2\bar{y}\mu-\bar{y}^2+2\bar{y}\bar{y})$$

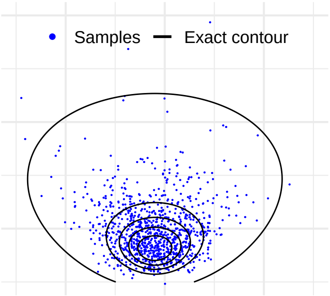
$$\sum_{i=1}^{n}(y_i-\bar{y})^2+n(\bar{y}-\mu)^2$$

Joint posterior

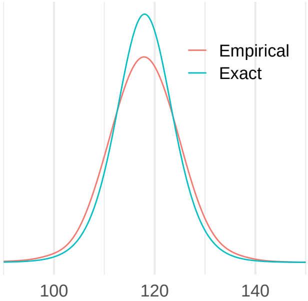


$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

Joint posterior



Marginal of mu

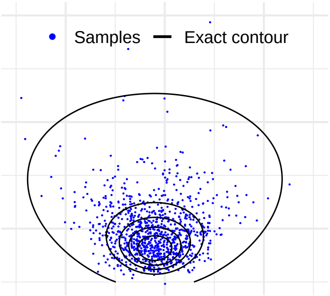


$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

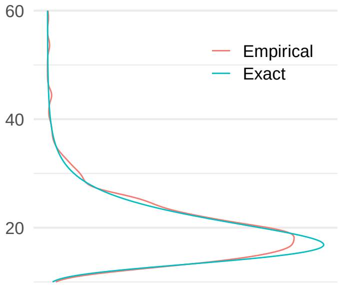
marginals

$$p(\mu \mid y) = \int p(\mu, \sigma \mid y) d\sigma$$

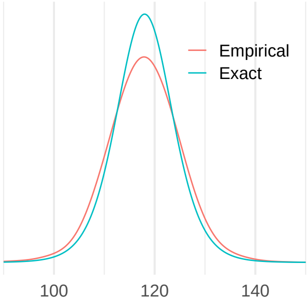
Joint posterior



Marginal of sigma



Marginal of mu



$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

marginals

$$p(\mu \mid y) = \int p(\mu, \sigma \mid y) d\sigma$$

$$p(\sigma \mid y) = \int p(\mu, \sigma \mid y) d\mu$$

Marginal posterior $p(\sigma^2 \mid y)$ (easier for σ^2 than σ)

$$p(\sigma^2 \mid y) \quad \propto \quad \int p(\mu, \sigma^2 \mid y) d\mu$$

Marginal posterior $p(\sigma^2 \mid y)$ (easier for σ^2 than σ)

$$\begin{aligned} p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\ &\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\mu \end{aligned}$$

Marginal posterior $p(\sigma^2 \mid y)$ (easier for σ^2 than σ)

$$
\begin{aligned}
p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\
&\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\mu \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \\
&\quad \int \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right) d\mu
\end{aligned}
$$

Marginal posterior $p(\sigma^2 \mid y)$ (easier for σ^2 than σ)

$$
\begin{aligned}
p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\
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&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \\
&\quad \int \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right) d\mu
\end{aligned}
$$

$$
\int \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right) d\theta = 1
$$

Marginal posterior $p(\sigma^2 \mid y)$ (easier for σ^2 than σ)

$$
\begin{aligned}
p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\
&\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\mu \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \\
&\quad \int \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right) d\mu \\
&\qquad\qquad \int \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right) d\theta = 1 \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \sqrt{2\pi\sigma^2/n}
\end{aligned}
$$

Marginal posterior $p(\sigma^2 \mid y)$ (easier for σ^2 than σ)

$$
\begin{aligned}
p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\
&\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\mu \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \\
&\qquad \int \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right) d\mu \\
&\qquad\qquad \int \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right) d\theta = 1 \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \sqrt{2\pi\sigma^2/n} \\
&\propto (\sigma^2)^{-(n+1)/2} \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right)
\end{aligned}
$$

Marginal posterior $p(\sigma^2 \mid y)$ (easier for σ^2 than σ)

$$\begin{aligned}
p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\
&\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\mu \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \\
&\quad \int \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right) d\mu
\end{aligned}$$

$$\int \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right) d\theta = 1$$

$$\begin{aligned}
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \sqrt{2\pi\sigma^2/n} \\
&\propto (\sigma^2)^{-(n+1)/2} \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right) \\
p(\sigma^2 \mid y) &= \text{Inv-}\chi^2(\sigma^2 \mid n-1, s^2)
\end{aligned}$$

Normal - non-informative prior

Known mean

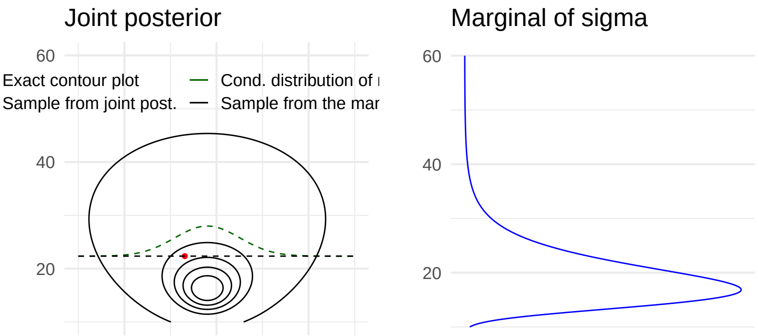
$$\sigma^2 \mid y \sim \text{Inv-}\chi^2(n, v)$$

$$\text{where} \quad v = \frac{1}{n}\sum_{i=1}^{n}(y_i - \theta)^2$$

Unknown mean

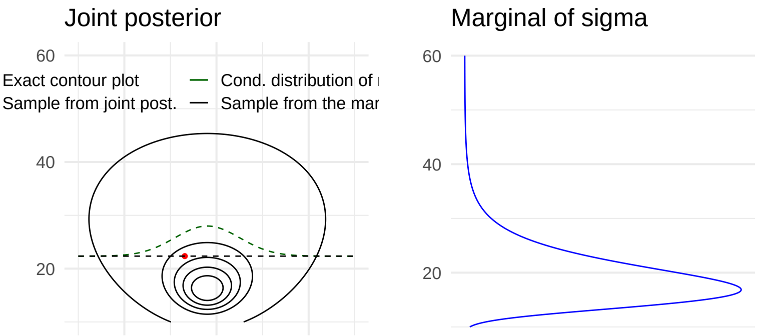
$$\sigma^2 \mid y \sim \text{Inv-}\chi^2(n-1, s^2)$$

$$\text{where} \quad s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$$



Factorization

$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$



Factorization

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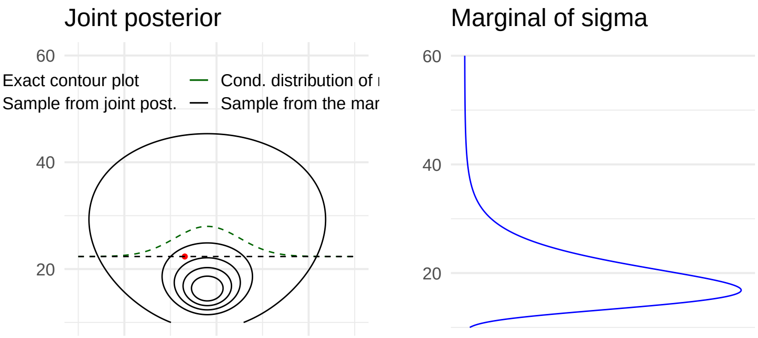
$$p(\sigma^2 \mid y) = \text{Inv-}\chi^2(\sigma^2 \mid n-1, s^2)$$

$$(\sigma^2)^{(s)} \sim p(\sigma^2 \mid y)$$



Factorization

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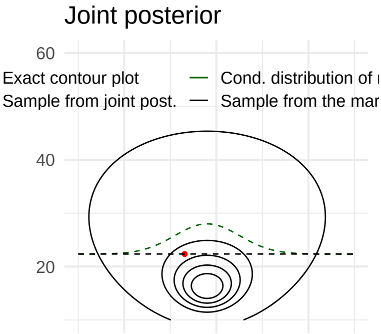
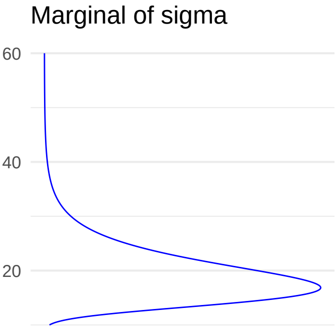
$$p(\mu \mid \sigma^2, y) = \text{N}(\mu \mid \bar{y}, \sigma^2/n)$$





Factorization

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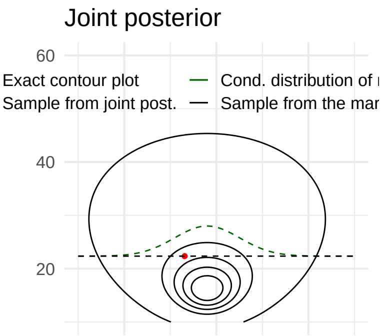
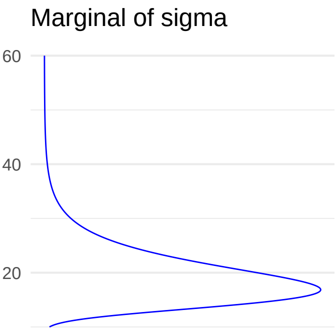
$$p(\sigma^2 \mid y) = \text{Inv-}\chi^2(\sigma^2 \mid n-1, s^2)$$

$$(\sigma^2)^{(s)} \sim p(\sigma^2 \mid y)$$

$$p(\mu \mid \sigma^2, y) = \text{N}(\mu \mid \bar{y}, \sigma^2/n) \propto \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right)$$





Factorization

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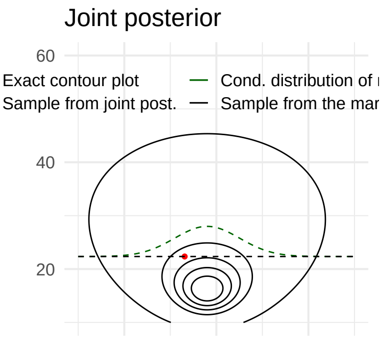
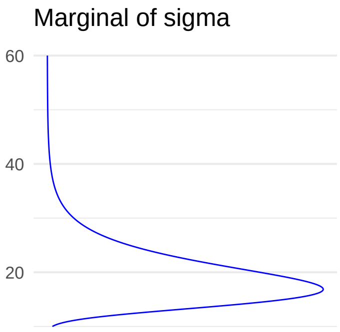
$$(\sigma^2)^{(s)} \sim p(\sigma^2 \mid y)$$

$$p(\mu \mid \sigma^2, y) = \text{N}(\mu \mid \bar{y}, \sigma^2/n)$$

$$\mu^{(s)} \sim p(\mu \mid \sigma^2, y)$$





Factorization

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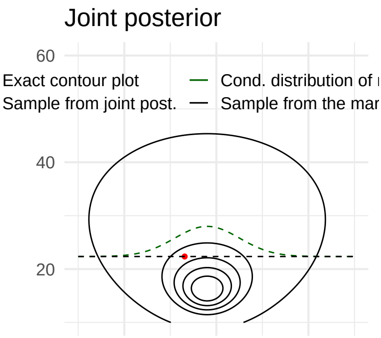
$$p(\sigma^2 \mid y) = \text{Inv-}\chi^2(\sigma^2 \mid n-1, s^2)$$

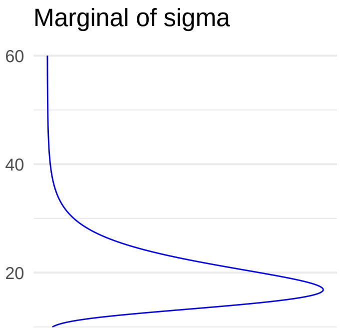
$$(\sigma^2)^{(s)} \sim p(\sigma^2 \mid y)$$

$$p(\mu \mid \sigma^2, y) = \text{N}(\mu \mid \bar{y}, \sigma^2/n)$$

$$\mu^{(s)} \sim p(\mu \mid \sigma^2, y)$$

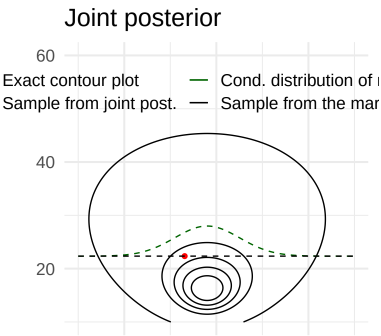
$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

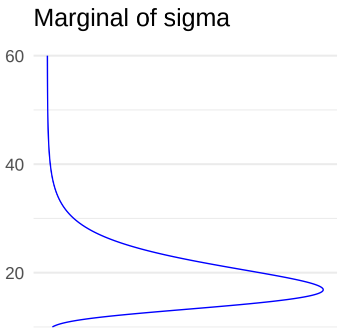




Factorization

$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$

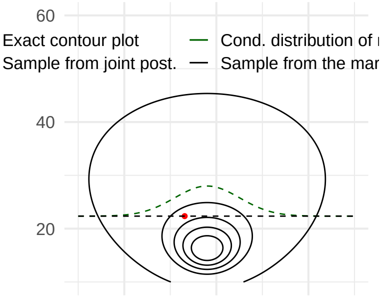




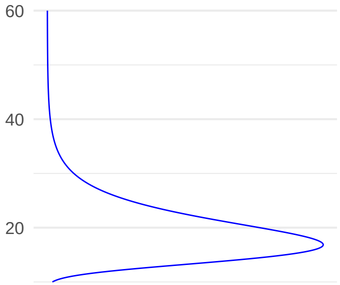
Factorization

$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$
$$(\sigma^2)^{(s)} \sim p(\sigma^2 \mid y)$$

Joint posterior



Marginal of sigma



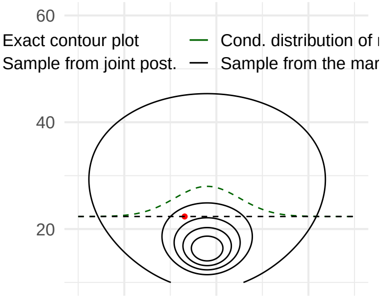
Factorization

$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$

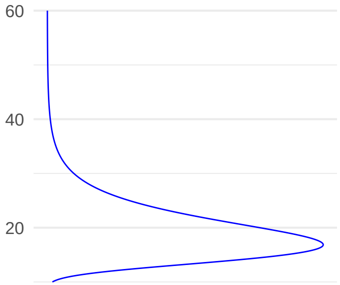
$$(\sigma^2)^{(s)} \sim p(\sigma^2 \mid y)$$

$$p(\mu \mid (\sigma^2)^{(s)}, y) = \mathrm{N}(\mu \mid \bar{y}, (\sigma^2)^{(s)}/n)$$

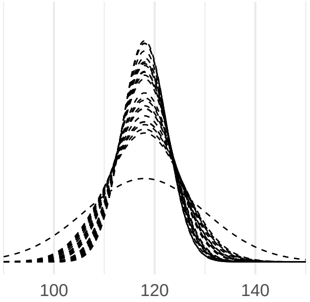
Joint posterior



Marginal of sigma



Cond distr of mu for 25 draws



Factorization

$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$

$$(\sigma^2)^{(s)} \sim p(\sigma^2 \mid y)$$

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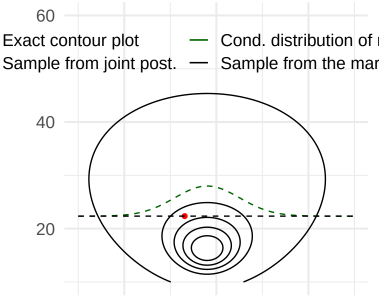
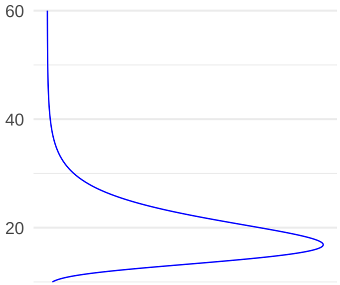
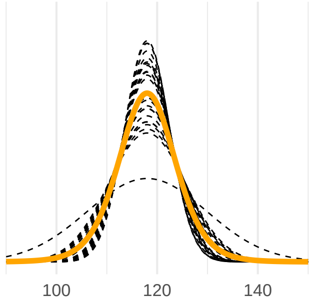
Joint posterior



Marginal of sigma



Cond distr of mu for 25 draws



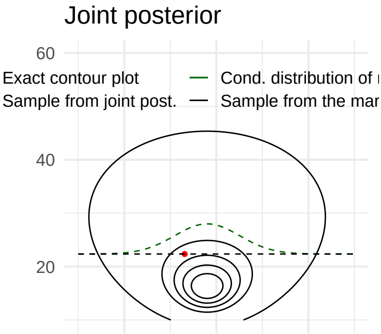
Factorization

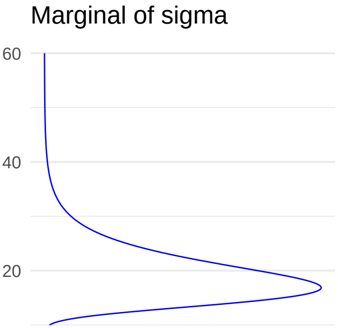
$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$

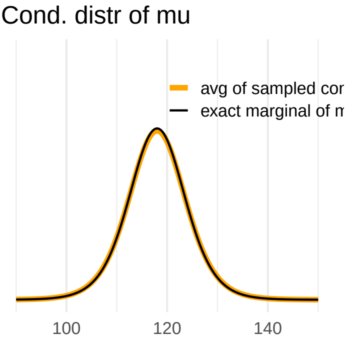
$$(\sigma^2)^{(s)} \sim p(\sigma^2 \mid y)$$

$$p(\mu \mid (\sigma^2)^{(s)}, y) = \mathrm{N}(\mu \mid \bar{y}, (\sigma^2)^{(s)}/n)$$

$$p(\mu \mid y) \approx \frac{1}{S} \sum_{s=1}^{S} \mathrm{N}(\mu \mid \bar{y}, (\sigma^2)^{(s)}/n)$$







Factorization

$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$

$$(\sigma^2)^{(s)} \sim p(\sigma^2 \mid y)$$

$$p(\mu \mid (\sigma^2)^{(s)}, y) = \mathrm{N}(\mu \mid \bar{y}, (\sigma^2)^{(s)}/n)$$

$$p(\mu \mid y) \approx \frac{1}{S} \sum_{s=1}^{S} \mathrm{N}(\mu \mid \bar{y}, (\sigma^2)^{(s)}/n)$$

Marginal posterior $p(\mu \mid y)$

$$p(\mu \mid y) = \int_0^\infty p(\mu, \sigma^2 \mid y) d\sigma^2$$

Marginal posterior $p(\mu \mid y)$

$$
\begin{aligned}
p(\mu \mid y) &= \int_0^\infty p(\mu, \sigma^2 \mid y) d\sigma^2 \\
&\propto \int_0^\infty \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\sigma^2
\end{aligned}
$$

Marginal posterior $p(\mu \mid y)$

$$p(\mu \mid y) = \int_0^\infty p(\mu, \sigma^2 \mid y) d\sigma^2$$
$$\propto \int_0^\infty \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\sigma^2$$

Transformation

$$A = (n-1)s^2 + n(\mu - \bar{y})^2$$

Marginal posterior $p(\mu \mid y)$

$$p(\mu \mid y) = \int_0^\infty p(\mu, \sigma^2 \mid y) d\sigma^2$$
$$\propto \int_0^\infty \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\sigma^2$$

Transformation

$$A = (n-1)s^2 + n(\mu - \bar{y})^2 \quad \text{and} \quad z = \frac{A}{2\sigma^2}$$

Marginal posterior $p(\mu \mid y)$

$$p(\mu \mid y) = \int_0^\infty p(\mu, \sigma^2 \mid y) d\sigma^2$$
$$\propto \int_0^\infty \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\sigma^2$$

Transformation

$$A = (n-1)s^2 + n(\mu - \bar{y})^2 \quad \text{and} \quad z = \frac{A}{2\sigma^2}$$

$$p(\mu \mid y) \propto A^{-n/2} \int_0^\infty z^{(n-2)/2} \exp(-z) dz$$

Marginal posterior $p(\mu \mid y)$

$$p(\mu \mid y) = \int_0^\infty p(\mu, \sigma^2 \mid y) d\sigma^2$$
$$\propto \int_0^\infty \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\sigma^2$$

Transformation

$$A = (n-1)s^2 + n(\mu - \bar{y})^2 \quad \text{and} \quad z = \frac{A}{2\sigma^2}$$

$$p(\mu \mid y) \propto A^{-n/2} \int_0^\infty z^{(n-2)/2} \exp(-z) dz$$

Recognize gamma integral $\Gamma(u) = \int_0^\infty x^{u-1} \exp(-x) dx$

Marginal posterior $p(\mu \mid y)$

$$p(\mu \mid y) = \int_0^\infty p(\mu, \sigma^2 \mid y) d\sigma^2$$

$$\propto \int_0^\infty \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y} - \mu)^2\right]\right) d\sigma^2$$

Transformation

$$A = (n-1)s^2 + n(\mu - \bar{y})^2 \quad \text{and} \quad z = \frac{A}{2\sigma^2}$$

$$p(\mu \mid y) \propto A^{-n/2} \int_0^\infty z^{(n-2)/2} \exp(-z) dz$$

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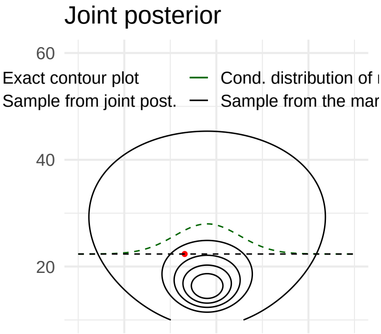

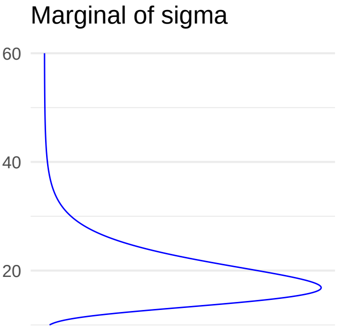

Predictive distribution for new $\tilde{y}$

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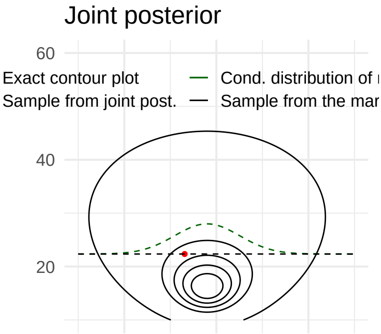

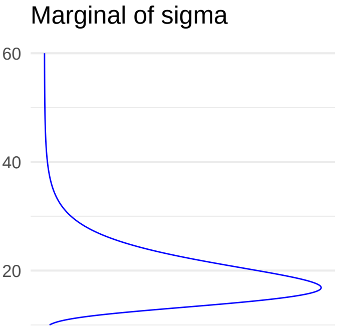

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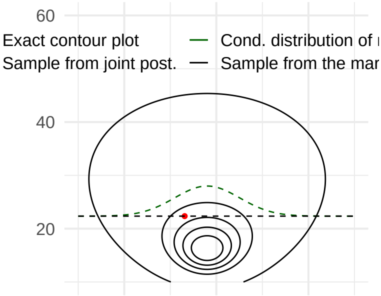
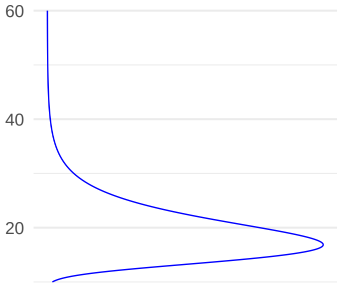


Marginal of sigma



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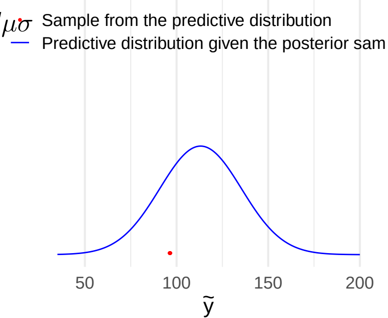

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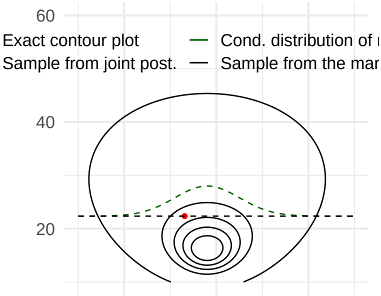

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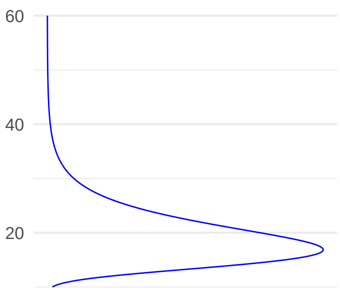

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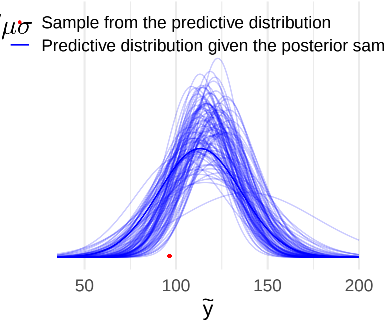

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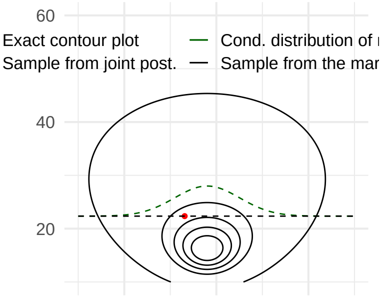

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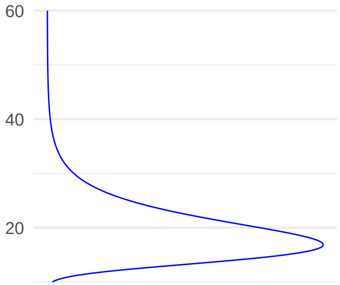

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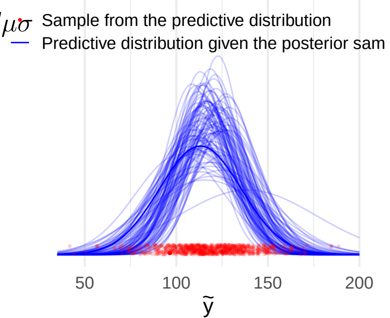

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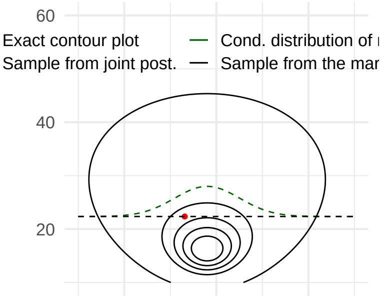

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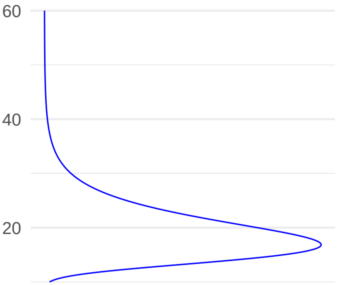

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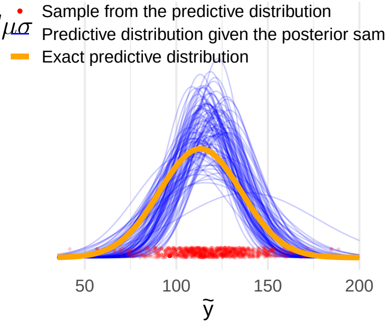

Normal - posterior predictive distribution

Posterior predictive distribution given known variance

$$p(\tilde{y} \mid \sigma^2, y) = \int p(\tilde{y} \mid \mu, \sigma^2) p(\mu \mid \sigma^2, y) d\mu$$

Normal - posterior predictive distribution

Posterior predictive distribution given known variance

$$
\begin{aligned}
p(\tilde{y} \mid \sigma^2, y) &= \int p(\tilde{y} \mid \mu, \sigma^2) p(\mu \mid \sigma^2, y) d\mu \\
&= \int \mathrm{N}(\tilde{y} \mid \mu, \sigma^2)\, \mathrm{N}(\mu \mid \bar{y}, \sigma^2/n) d\mu
\end{aligned}
$$

Normal - posterior predictive distribution

Posterior predictive distribution given known variance

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$$
p(\tilde{y} \mid y) = t_{n-1}(\tilde{y} \mid \bar{y}, (1 + \tfrac{1}{n}) s^2)
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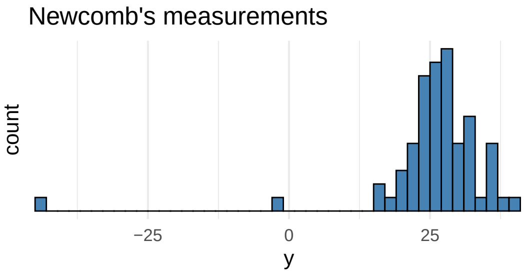
Simon Newcomb's light of speed experiment in 1882

Newcomb measured ($n = 66$) the time required for light to travel from his laboratory on the Potomac River to a mirror at the base of the Washington Monument and back, a total distance of 7422 meters.



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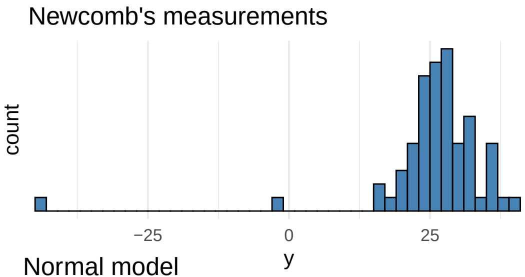

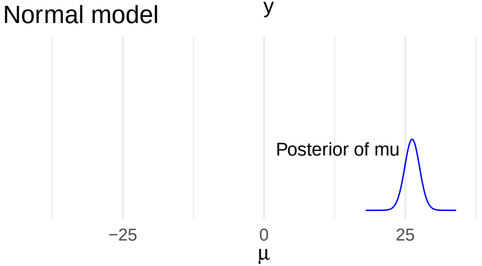

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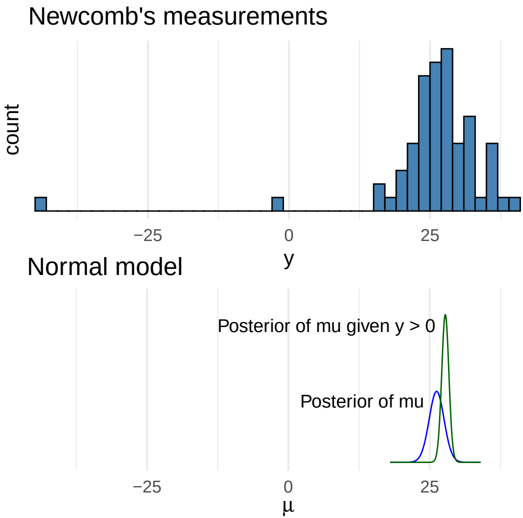

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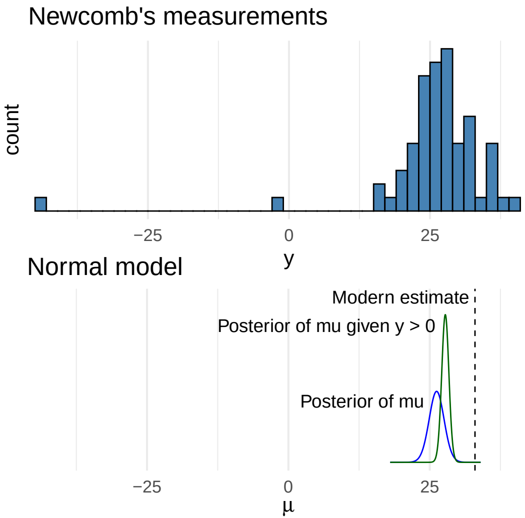

Normal - conjugate prior

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which can be written as

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$$p(\mu, \sigma^2) = \text{N-Inv-}\chi^2(\mu_0, \sigma_0^2/\kappa_0; \nu_0, \sigma_0^2)$$

- μ and σ^2 are a priori dependent
  - if σ^2 is large, then μ has wide prior

Normal - conjugate prior

Joint posterior (exercise 3.9)

$$p(\mu, \sigma^2 \mid y) = \text{N-Inv-}\chi^2(\mu_n, \sigma_n^2/\kappa_n; \nu_n, \sigma_n^2)$$

where

$$\begin{aligned}
\mu_n &= \frac{\kappa_0}{\kappa_0 + n}\mu_0 + \frac{n}{\kappa_0 + n}\bar{y} \\
\kappa_n &= \kappa_0 + n \\
\nu_n &= \nu_0 + n \\
\nu_n\sigma_n^2 &= \nu_0\sigma_0^2 + (n-1)s^2 + \frac{\kappa_0 n}{\kappa_0 + n}(\bar{y} - \mu_0)^2
\end{aligned}$$

Comparison of means of two normals

- The difference of two normally distributed variables is normally distributed
- The difference of two t distributed variables with different variances and degrees of freedom doesn't have an easy form
  - easy to sample from the two distributions, and obtain samples of the differences

$$\delta^{(s)} = \mu_1^{(s)} - \mu_2^{(s)}$$

Multivariate normal

- Observation model

$$p(y \mid \mu, \Sigma) \propto \mid \Sigma \mid^{-1/2} \exp\left(-\frac{1}{2}(y-\mu)^T\Sigma^{-1}(y-\mu)\right),$$

- BDA3 p. 72–
- New recommended LKJ-prior mentioned in Appendix A, see more in Stan manual

Multivariate normal

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- BDA3 p. 72–
- New recommended LKJ-prior mentioned in Appendix A, see more in Stan manual
- Gaussian process and Gaussian Markov random field models are in practice computed with multivariate normals
    - GPs in BDA3 Chapter 21, and a course in spring
    - GPs and GMRFs often used also as priors for latent functions and combined with non-normal observation models

Normal linear regression

- $y_i \sim \mathrm{N}(\alpha + \beta x_i, \sigma^2), \quad i = 1, \ldots, N$

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- with unknown prior scales and σ^2, numerical integration needed
- more in BDA3 Chapter 14 (not part of the course) and Regression and Other Stories book

Scale mixture of normals

- Many useful distributions can be presented as scale mixture of normals, e.g.
    - Student's t
    - Cauchy
    - Double exponential aka Laplace
    - Horseshoe
    - R2-D2

Multinomial model for categorical data

- Extension of binomial
- Observation model

$$p(y \mid \theta) \propto \prod_{j=1}^{k} \theta_j^{y_j},$$

- BDA3 p. 69–

Generalized linear model (GLM)

- $y_i \sim p(g^{-1}(\alpha + \beta x_i), \phi), \quad i = 1, \ldots, N$
    - where p is non-normal (in original definition in exponential family)
    - and g is a link function

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- Bioassay analysis is used as an example

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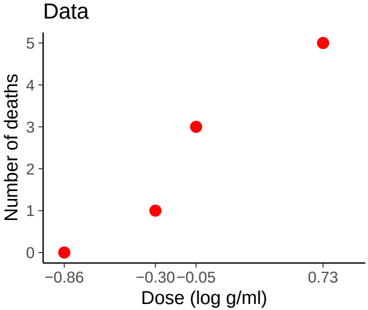
- Bioassay analysis is used as an example
- More in BDA3 Chapter 16 and Regression and other stories book

Bioassay

Dose, x_i (log g/ml)	Number of animals, n_i	Number of deaths, y_i
-0.86	5	0
-0.30	5	1
-0.05	5	3
0.73	5	5



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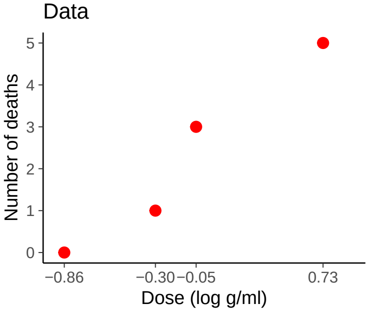

Find out lethal dose 50% (LD50)

- used to classify how hazardous chemical is
- 1984 EEC directive has 4 levels (see the chapter notes)

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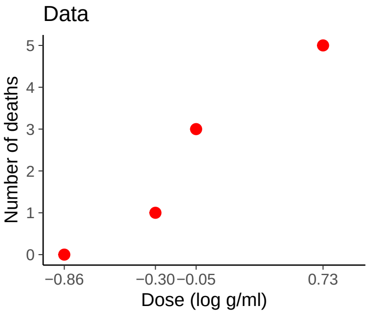

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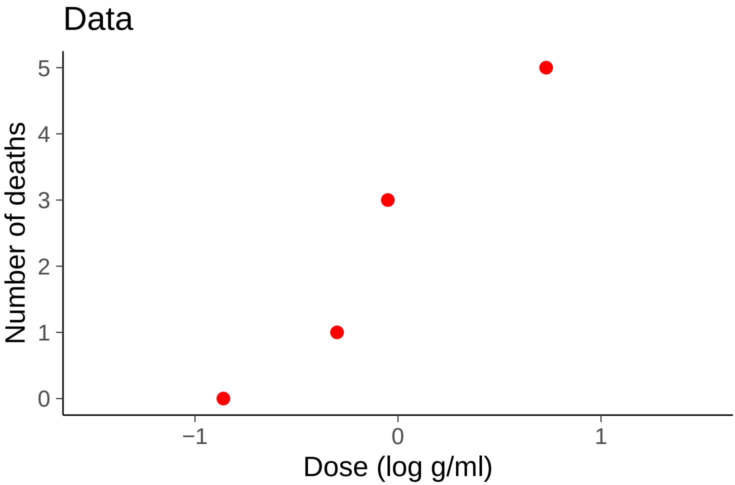
Bayesian methods help to

- reduce the number of animals needed
- easy to make sequential experiment and stop as soon as desired accuracy is obtained

Bioassay



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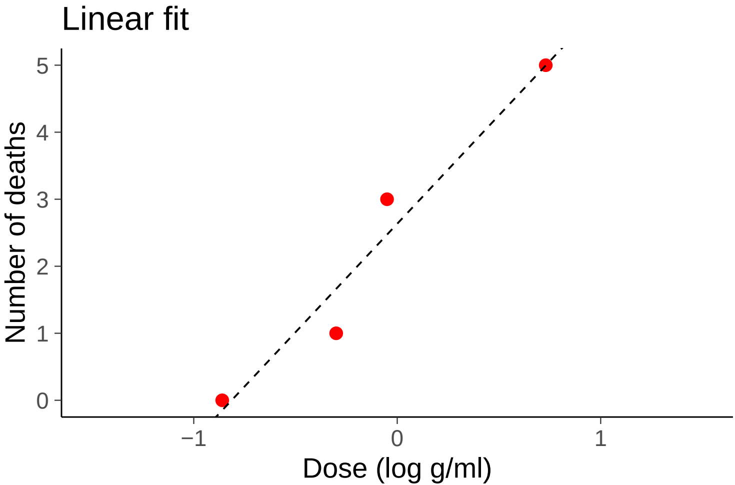

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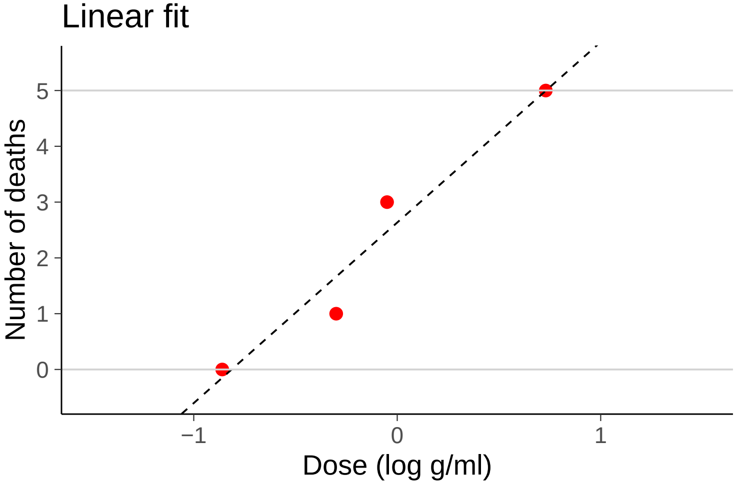

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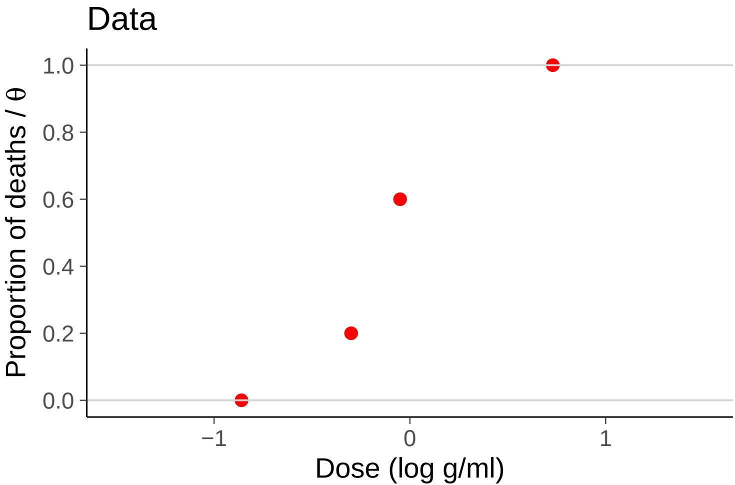

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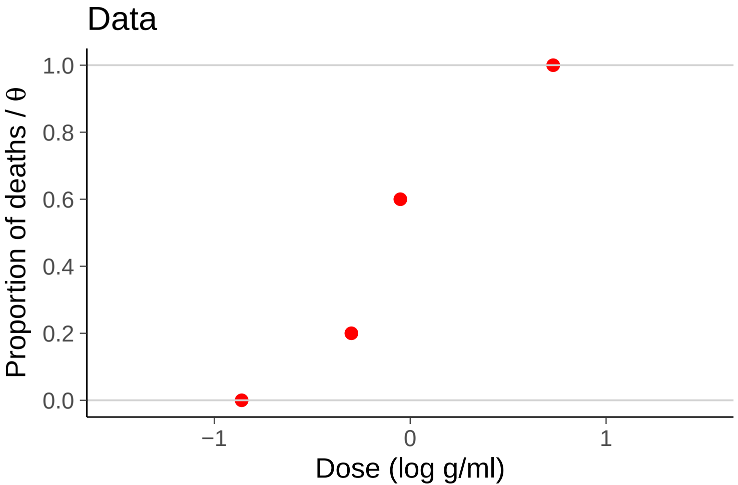
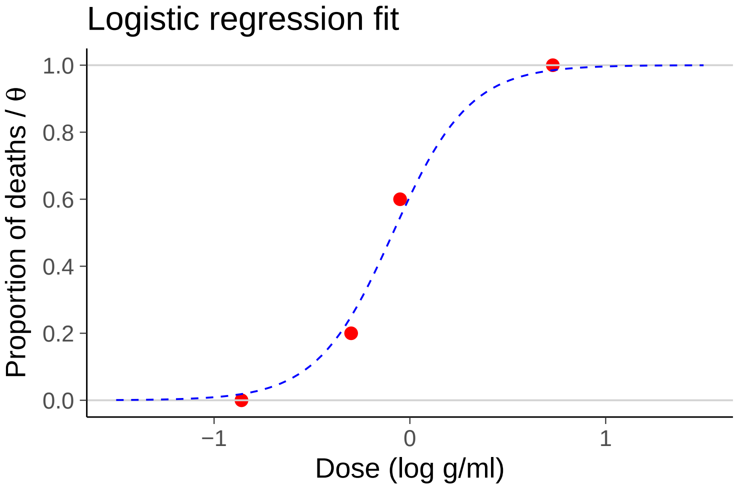


Binomial model

$$y_i \mid \theta_i \sim \text{Bin}(\theta_i, n_i)$$

Bioassay



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Bioassay

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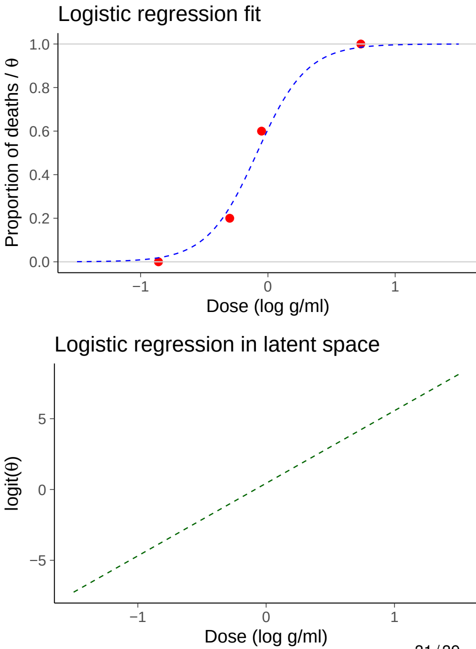
$$= \alpha + \beta x_i$$

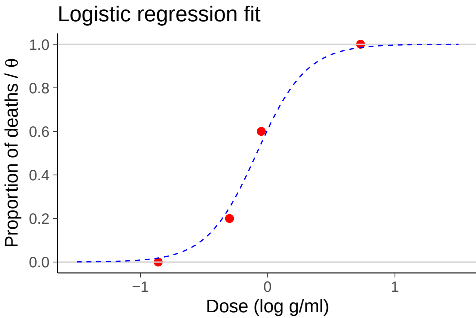


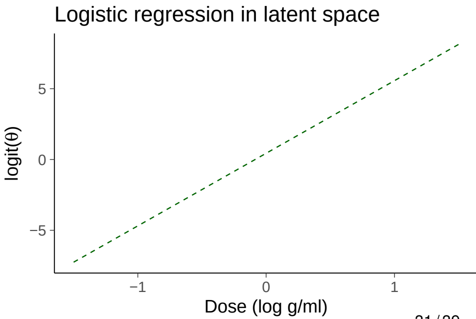
Bioassay

$$y_i \mid \theta_i \sim \text{Bin}(\theta_i, n_i)$$

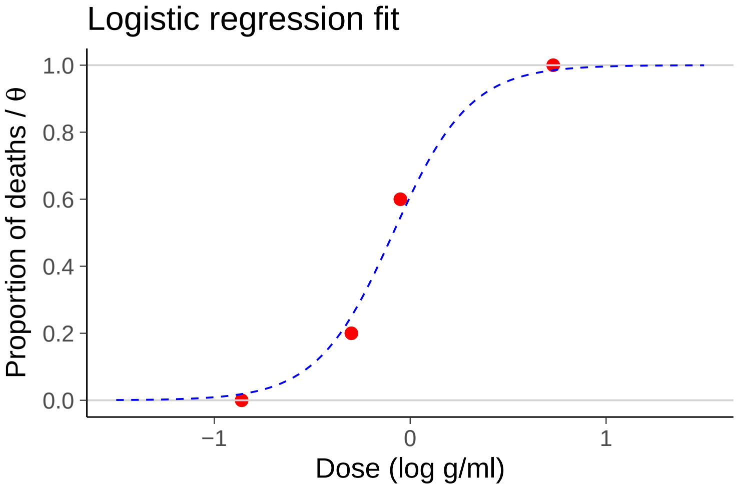
$$\text{logit}(\theta_i) = \log\left(\frac{\theta_i}{1-\theta_i}\right) = \alpha + \beta x_i$$

$$\theta_i = \frac{1}{1+\exp(-(\alpha + \beta x_i))}$$

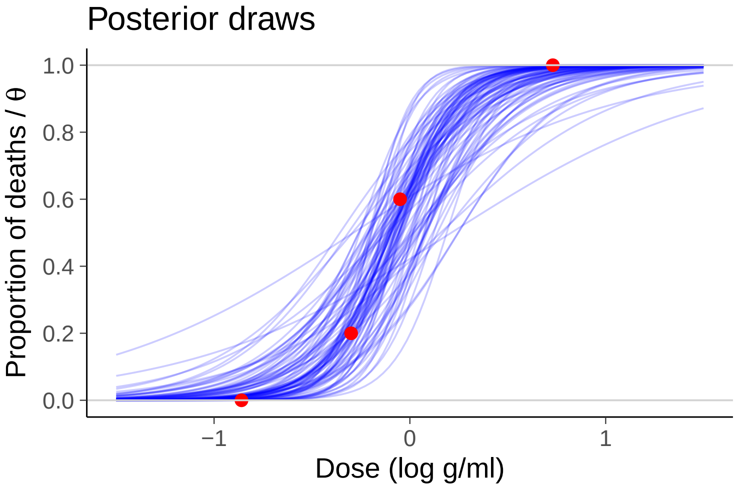




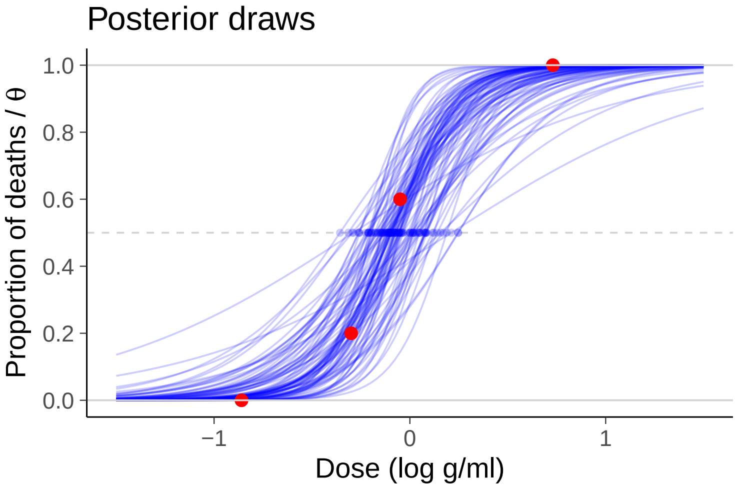
Bioassay


Logistic regression fit
Proportion of deaths / θ
1.0
0.8
0.6
0.4
0.2
0.0
−1
0
1
Dose (log g/ml)

Bioassay

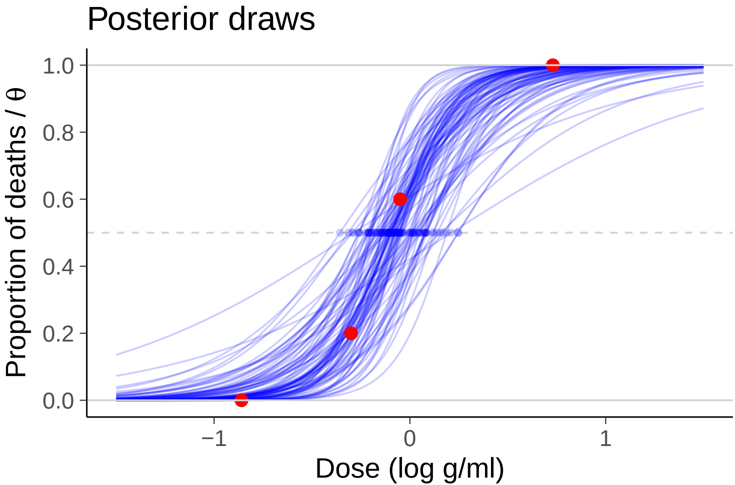

Posterior draws
Proportion of deaths / θ
1.0
0.8
0.6
0.4
0.2
0.0
−1
0
1
Dose (log g/ml)

Bioassay



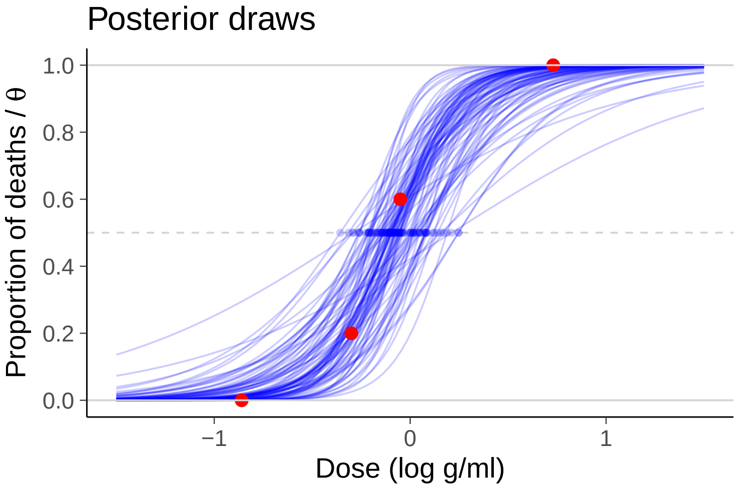
LD50: $\mathrm{E}\left(\frac{y}{n}\right) = \mathrm{logit}^{-1}(\alpha + \beta x) = 0.5$

Bioassay



LD50: $\mathrm{E}\left(\frac{y}{n}\right) = \mathrm{logit}^{-1}(\alpha + \beta x) = 0.5 \quad \Rightarrow \quad x_{\mathrm{LD50}} = -\alpha/\beta$

Bioassay



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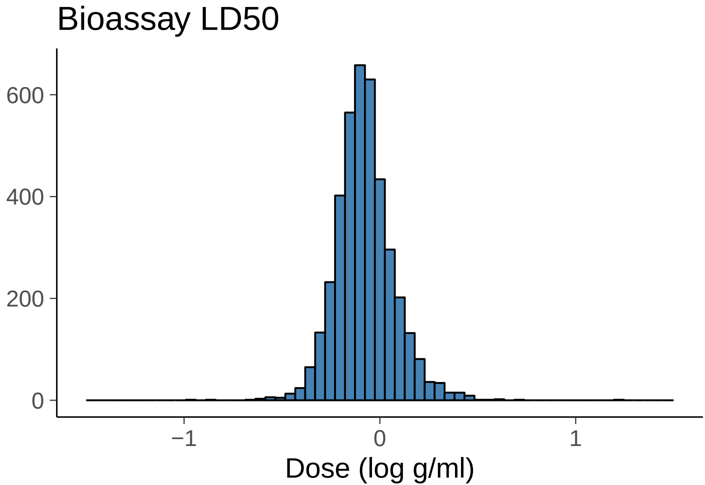
$$x^{(s)}_{\mathrm{LD50}} = -\alpha^{(s)}/\beta^{(s)}$$

Bioassay



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$$x_{\text{LD50}}^{(s)} = -\alpha^{(s)}/\beta^{(s)}$$

Bioassay posterior

Binomial model

$$y_i \mid \theta_i \sim \mathrm{Bin}(\theta_i, n_i)$$

Link function

$$\mathrm{logit}(\theta_i) = \alpha + \beta x_i$$

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Likelihood

$$p(y_i \mid \alpha, \beta, n_i, x_i) \propto \theta_i^{y_i}[1 - \theta_i]^{n_i - y_i}$$

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$$p(y_i \mid \alpha, \beta, n_i, x_i) \propto [\mathrm{logit}^{-1}(\alpha + \beta x_i)]^{y_i}[1 - \mathrm{logit}^{-1}(\alpha + \beta x_i)]^{n_i - y_i}$$

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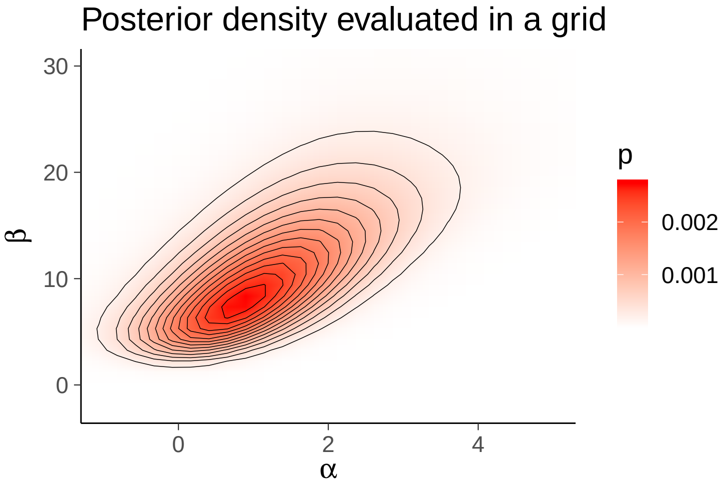
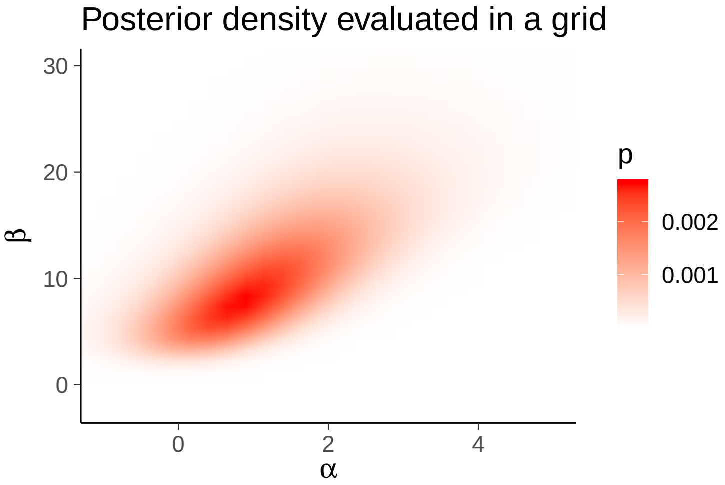
Posterior (with uniform prior on α, β)

$$p(\alpha, \beta \mid y, n, x) \propto p(\alpha, \beta) \prod_{i=1}^{n} p(y_i \mid \alpha, \beta, n_i, x_i)$$

Bioassay

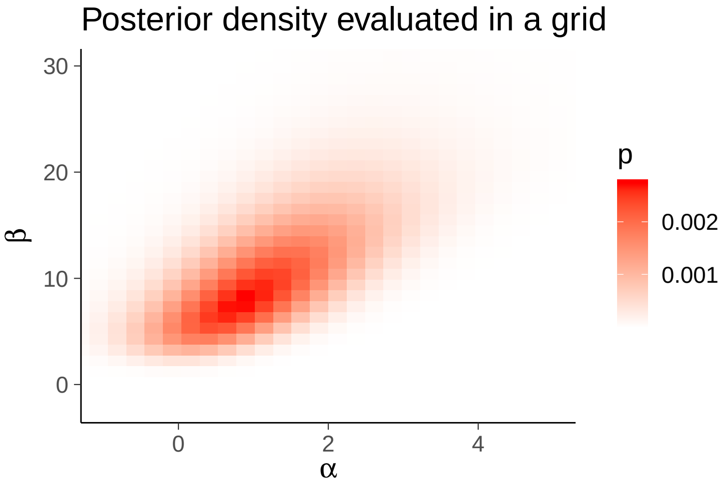

Posterior density evaluated in a grid
30
20
10
0
β
0
2
4
α
p
0.002
0.001

Bioassay



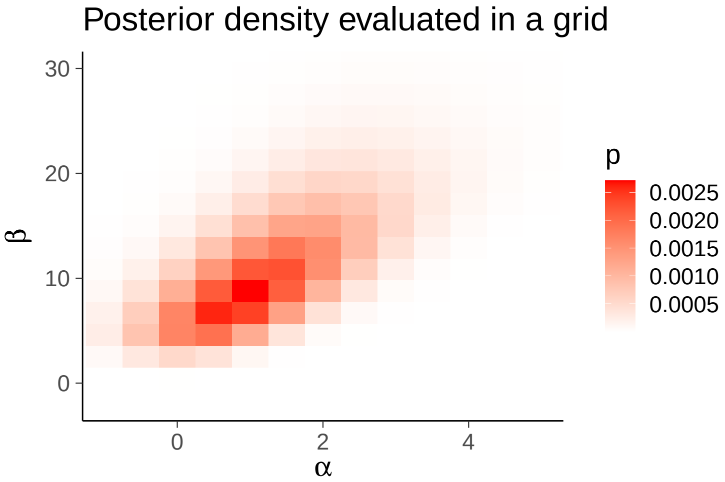
Density evaluated in grid, but plotted using interpolation

Bioassay



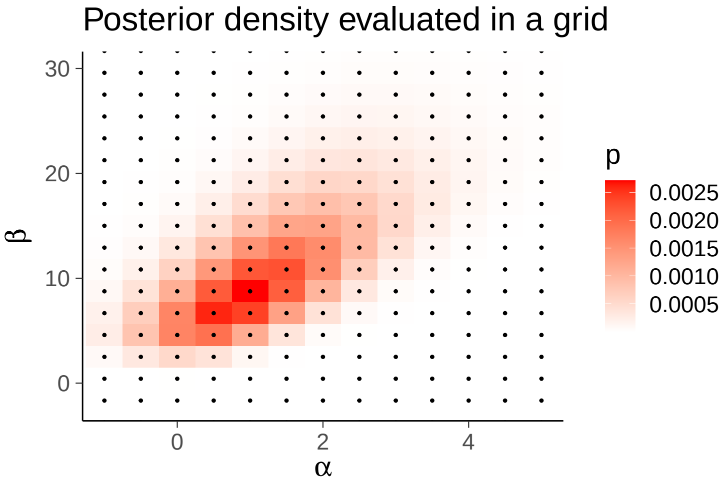
Density evaluated in grid, and plotted without interpolation

Bioassay



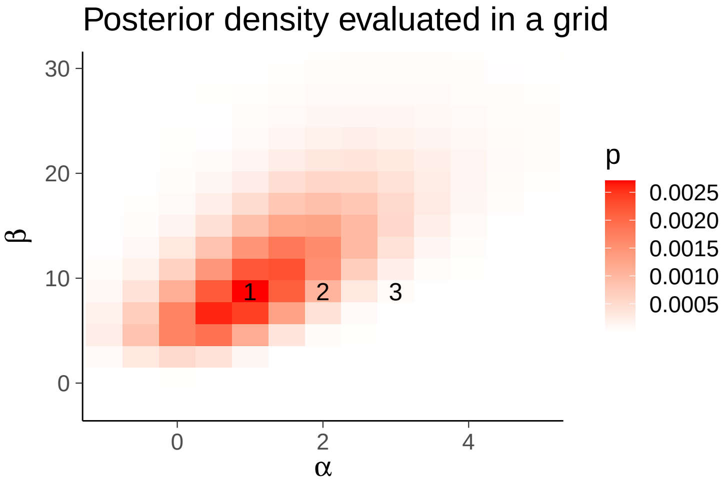
Density evaluated in a coarser grid

Bioassay



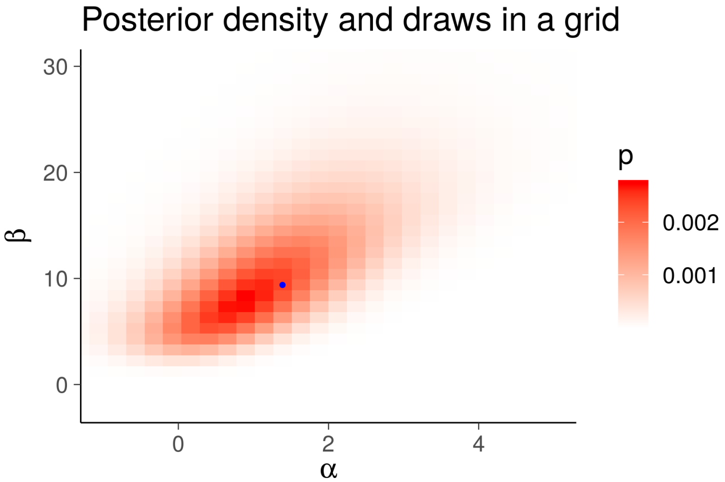
- Approximate the density as piecewise constant function
- Evaluate density in a grid over some finite region
- Density times cell area gives probability mass in each cell

Bioassay



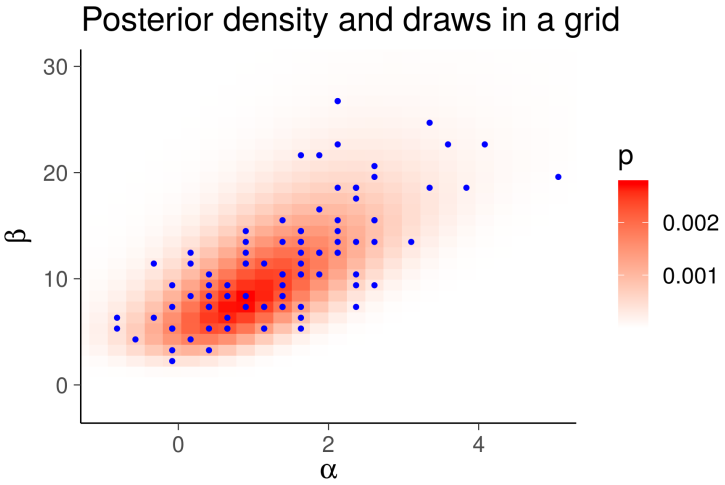
- Densities at 1, 2, and 3: 0.0027 0.0010 0.0001
- Probabilities of cells 1, 2, and 3: 0.0431 0.0166 0.0010
- Probabilities of cells sum to 1

Bioassay



- Sample according to grid cell probabilities

Bioassay



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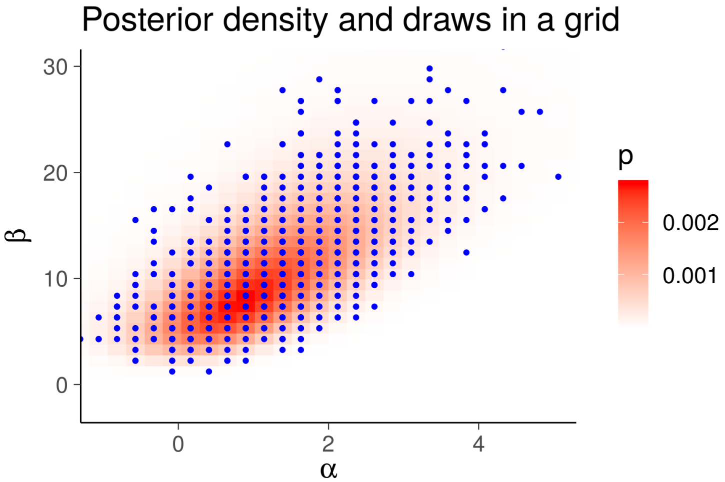
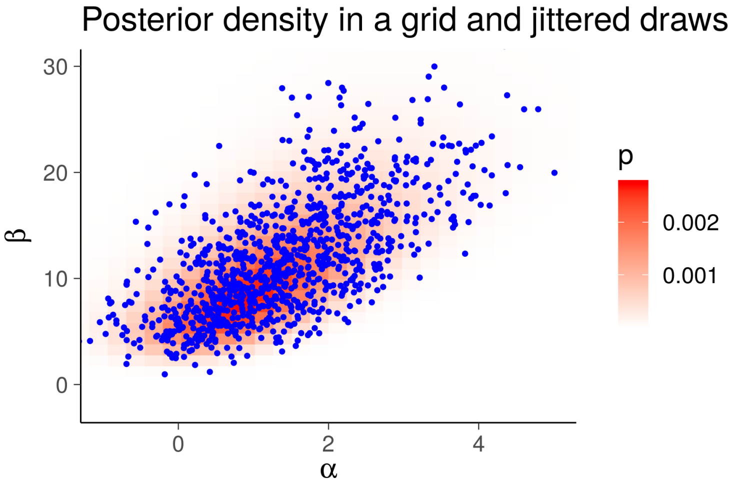


- Sample according to grid cell probabilities
- Several draws can be from the same grid cell

Bioassay



- Jitter can be added to improve visualization

Grid sampling

- Draws can be used to estimate expectations, for example

$$E[x_{\mathrm{LD50}}] = E[-\alpha/\beta] \approx \frac{1}{S}\sum_{s=1}^{S} -\frac{\alpha^{(s)}}{\beta^{(s)}}$$

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$$\mathrm{E}[-\alpha/\beta] \approx \sum_{t=1}^{T} -\frac{\alpha^{(t)}}{\beta^{(t)}} w_{\mathrm{cell}}^{(t)},$$

where $w_{\mathrm{cell}}^{(t)}$ is the normalized probability of a grid cell t, and $\alpha^{(t)}$ and $\beta^{(t)}$ are center locations of grid cells

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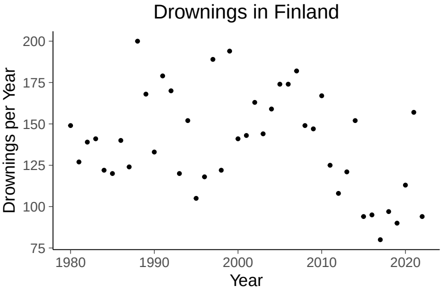
- Grid sampling gets computationally too expensive in high dimensions

Example GLM

Count of deaths, y_i	Year
149	1980
127	1981
139	1982
⋮	⋮
157	2021
94	2022



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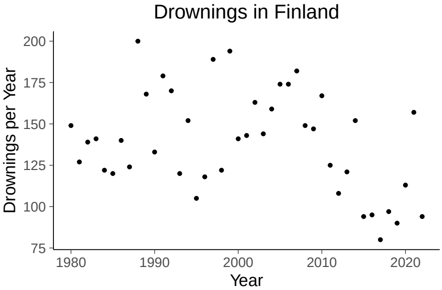

Swimming is popular in Finland, but also hazardous

- On average $\sim$140 drownings per year
- Finnish government has invested in measures for reducing deaths
- Recent narrative based on effectiveness of education

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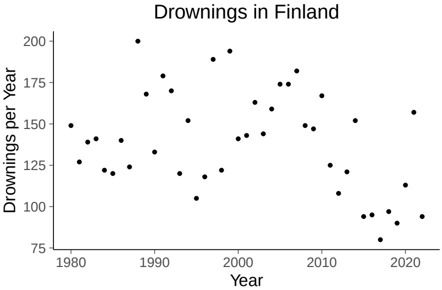

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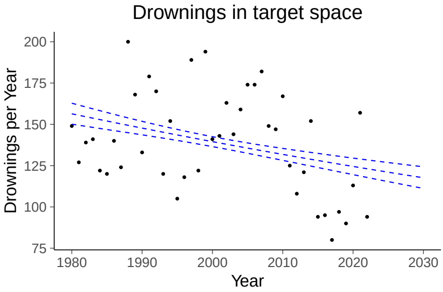
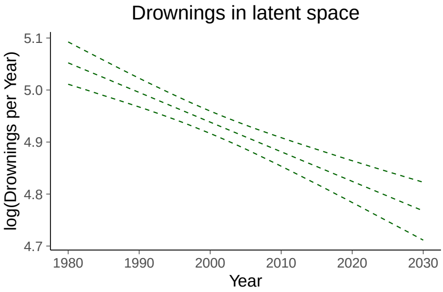
Bayesian methods help

- Describe trends over time
- Evaluate uncertainty

Example GLM: Poisson Linear Model

$$y_i \mid \mu_i \sim \text{Poisson}(\mu_i)$$
$$\mu_i = e^{\alpha + \beta x_i}$$





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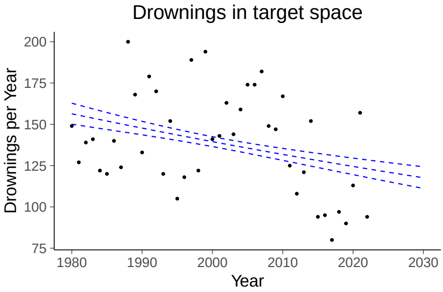

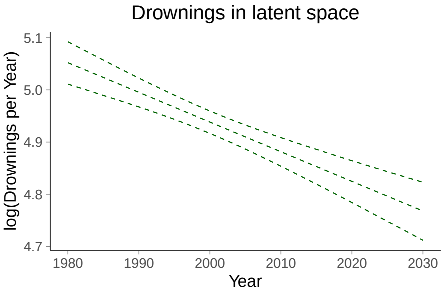

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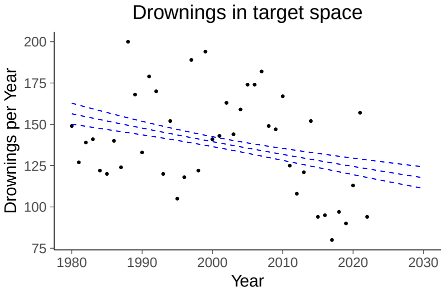
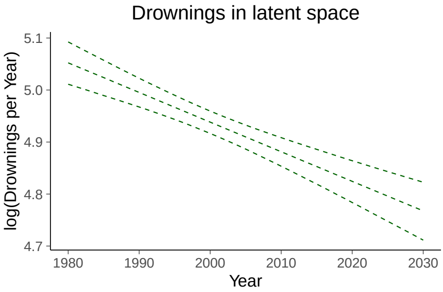
Alternatively :

$$y_i \mid \mu_i, \phi \sim \text{Neg-bin}(y_i|\mu_i, \phi)$$





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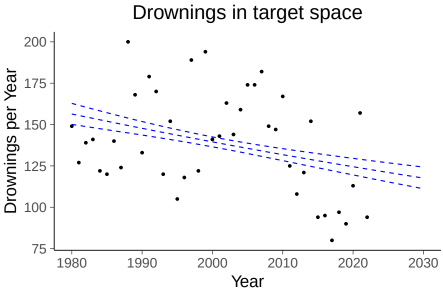
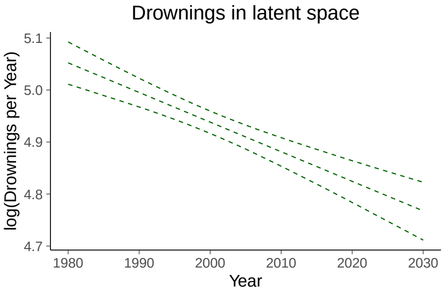


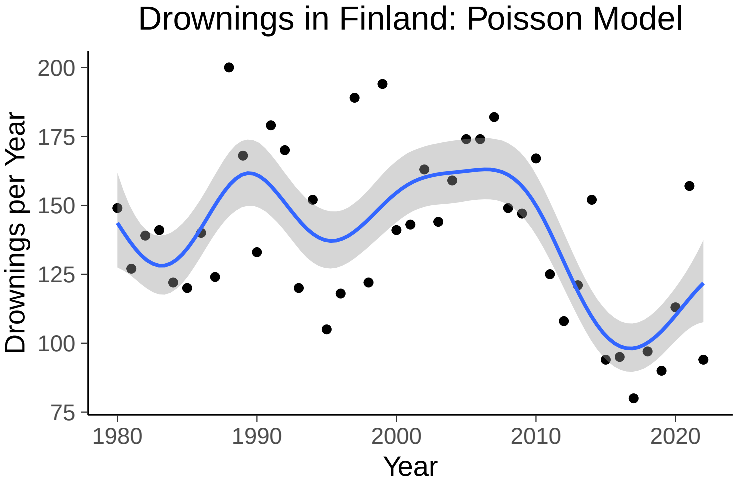
Alternatively :

$$y_i \mid \mu_i, \phi \sim \text{Neg-bin}(y_i|\mu_i, \phi)$$

$$\text{Neg-bin}(y_i \mid \mu_i, \phi) = \frac{\Gamma(y_i+\phi)}{y_i!\Gamma(\phi)}\left(\frac{\mu_i}{\mu_i+\phi}\right)^{y_i}\left(\frac{\phi}{y_i+\phi}\right)^{\phi}$$



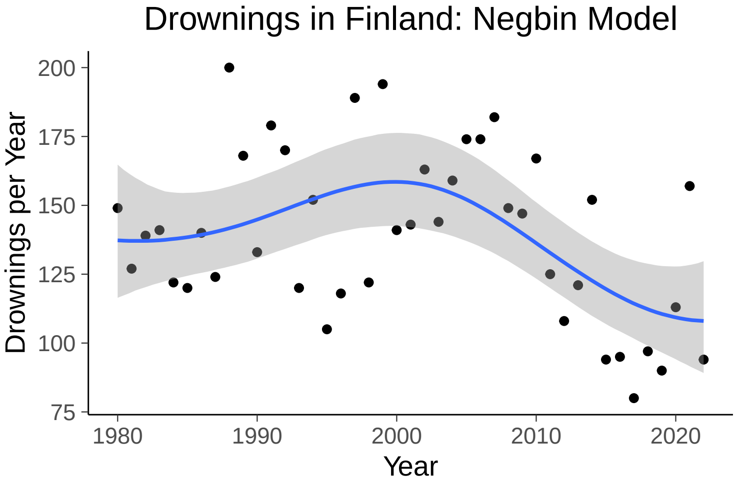
Example GLM: Gaussian Process Models



$$y_i \mid \mu_i \sim \text{Poisson}(\mu_i)$$

$$\mu_i \sim e^{f_i}, \; f \sim \text{multi normal}(0, \text{k}(\text{Year}))$$

Example GLM: Gaussian Process Models



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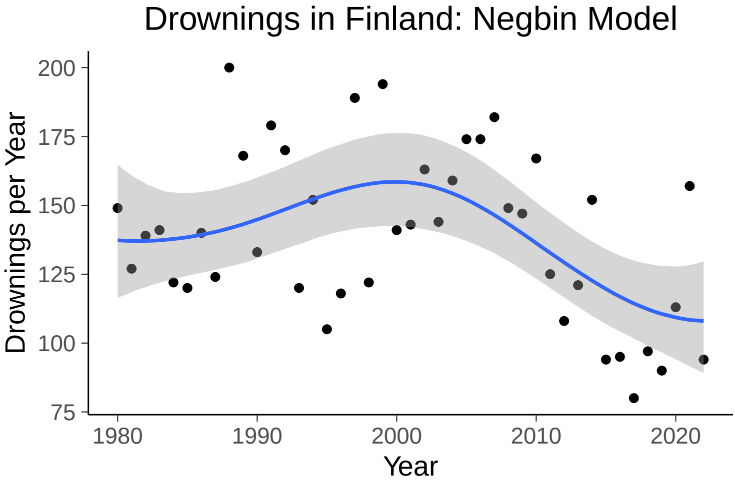
Example GLM: Gaussian Process Models



- Clear overdispersion
- Trend interpretations shouldn't be based on one observation

Thinking counts

- For simplicity of exposition, we often start learning with normal observation models
- But we observe count data on a daily basis
- Very relevant in industry (number of sold products, ad views, customer count, etc.)
- Can you think of such examples from the class room?
  - Think of how many students attend BDA lectures over the course
  - Number of students who report getting sick over time until Christmas
  - Number of dropouts
  - Would you expect overdispersion?